EXISTENCE AND STABILITY OF SOLITARY WAVES OF AN M-COUPLED NONLINEAR SCHRÖDINGER SYSTEM

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Abstract. In this paper, the existence and stability results for ground state solutions of an m-coupled nonlinear Schrödinger system

\[ i \frac{\partial}{\partial t} u_j + \frac{\partial^2}{\partial x^2} u_j + \sum_{i=1}^{m} b_{ij} |u_i|^p |u_j|^{p-2} u_j = 0, \]

are established, where \(2 \leq m, 2 \leq p < 3\) and \(u_j\) are complex-valued functions of \((x, t) \in \mathbb{R}^2, j = 1, \cdots, m\) and \(b_{ij}\) are positive constants satisfying \(b_{ij} = b_{ji}\). In contrast with other methods used before to establish existence and stability of solitary wave solutions where the constraints of the variational minimization problem are related to one another, our approach here characterizes ground state solutions as minimizers of an energy functional subject to independent constraints. The set of minimizers is shown to be orbitally stable and further information about the structure of the set is given in certain cases.

1. Introduction

The nonlinear Schrödinger (NLS) equation

\[ iu_t + u_{xx} + |u|^{p-2}u = 0, \]  

(1.1)

where \(2 < p\) and \(u\) is a complex function of \((x, t) \in \mathbb{R}^2\) arises in several applications. The equation describes evolution of small amplitude, slowly varying wave packets in a nonlinear media [3]. Indeed, it has been derived in such diverse fields as deep water waves [22], plasma physics [23], nonlinear optical fibers [8, 9], magneto-static spin waves [24], to name a few. The coupled nonlinear Schrödinger (CNLS) system

\[ i \frac{\partial}{\partial t} u_j + \frac{\partial^2}{\partial x^2} u_j + \sum_{i=1}^{m} b_{ij} |u_i|^p |u_j|^{p-2} u_j = 0, \]  

(1.2)

where \(2 \leq p < 3, m \geq 2\) and \(u_j\) are complex-valued functions of \((x, t) \in \mathbb{R}^2, j = 1, 2, \cdots, m\) and \(b_{ij} \in \mathbb{R}\), arises physically under conditions similar to those described by (1.1) when there are m-wave trains moving with nearly the same group velocities [19, 21]. The CNLS system also models physical systems whose fields have more than one components: for example, in optical fibers and waveguides, the propagating electric field has two components that are transverse to the direction of propagation. These types of systems also arise from physical models in nonlinear optics and in Bose-Einstein condensates for multi-species condensates (i.e., [14, 20] and references therein). Readers are referred to the works [8, 9, 3, 14, 20, 22, 23] for the derivation as well as applications of the system (1.2). With coupling effects in the system, some new features of the solutions structure arise that do not exist in the single equation (1.1).

Notation. For \(1 \leq p \leq \infty\), we denote by \(L^p = L^p(\mathbb{R})\) the space of all complex-valued measurable functions \(f\) on \(\mathbb{R}\) for which the norm \(\|f\|_p = (\int_{\mathbb{R}} |f|^p dx)^{\frac{1}{p}}\) is finite for \(1 \leq p < \infty\), and \(\|f\|_\infty\) is the essential supremum of \(|f|\) on \(\mathbb{R}\). The space \(H^1_\mathcal{C}(\mathbb{R})\) is the usual Sobolev space consisting of measurable functions such that both \(f\) and \(f_x\) are in \(L^2\) and we define the space \(X_j\) to be the \(j\)-times Cartesian product \(X_j = H^1_\mathcal{C}(\mathbb{R}) \times H^1_\mathcal{C}(\mathbb{R}) \times \cdots \times H^1_\mathcal{C}(\mathbb{R})\). If \(T > 0\) and \(Y\) is any Banach space, we denote by \(C([0, T], Y)\) the Banach space of continuous maps \(f : [0, T] \rightarrow Y\), with norms given by \(\|f\|_{C([0, T], Y)} = \sup_{[0, T]} \|f(t)\|_Y\).

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Review. Global well-posedness for the system (1.2) follows from [6] (see also [15]). Precisely, it was proved that for any initial data \((u_1(x,0), u_2(x,0), \cdots, u_m(x,0)) \in X_m\), there exists a unique solution \((u_1(x,t), u_2(x,t), \cdots, u_m(x,t))\) of (1.2) in \(C(\mathbb{R}; X_m)\) emanating from \((u_1(x,0), u_2(x,0), \cdots, u_m(x,0))\), and \((u_1(x,t), u_2(x,t), \cdots, u_m(x,t))\) satisfies
\[
E(u_1(\cdot,t), u_2(\cdot,t), \cdots, u_m(\cdot,t)) = E(u_1(\cdot,0), u_2(\cdot,0), \cdots, u_m(\cdot,0)),
\]
\[
Q(u_j(\cdot,t)) = Q(u_j(\cdot,0)),
\]
where \(E\) and \(Q\) are the following conserved quantities
\[
E(u_1, u_2, \cdots, u_m) = \int_\mathbb{R} \left( \sum_{j=1}^m \left( \frac{d}{dx} u_j(x,t) \right)^2 - \frac{1}{p} \sum_{i,j=1}^m b_{ij} |u_i(x,t)|^p |u_j(x,t)|^p \right) dx,
\]
\[
Q(u_j) = \int_\mathbb{R} |u_j(x,t)|^2 dx,
\]
for \(j = 1, 2, \cdots, m\).

Solitary-wave solutions of (1.2) are solutions of the form
\[
 u_j(x,t) = e^{i[(\omega_j-\sigma)x + \sigma t + m_j]} \phi_j(x - 2\sigma t) \tag{1.3}
\]
for \(j = 1, 2, \cdots, m\), and \(\omega_j, m_j \in \mathbb{R}\) and \(\phi_j : \mathbb{R} \to \mathbb{R}\) are functions that vanish at infinity in the sense that \(\phi_j \in H^1_\sigma(\mathbb{R})\). An important special case arises when \(m_j = \sigma = 0\). These special solutions
\[
 u_j(x,t) = e^{i\omega_j t} \phi_j(x) \tag{1.4}
\]
are often referred to as standing waves. It is easy to see that standing wave is a solution of (1.2) if and only if \((\phi_1, \phi_2, \cdots, \phi_m)\) solves the system of ordinary differential equations
\[
 -\phi_j'' + \omega_j \phi_j = \sum_{i=1}^m b_{ij} |\phi_i|^p |\phi_j|^{p-2} \phi_j \tag{1.5}
\]
for \(j = 1, 2, \cdots, m\). Equivalently, standing wave is a solution of (1.2) if and only if \((u_1, u_2, \cdots, u_m)\) is a critical point for the functional \(E(u_1, u_2, \cdots, u_m)\), subject to the \(m\) constraints \(Q(u_j)\) being held constants. If \((u_1, u_2, \cdots, u_m)\) is not only a critical point but in fact the global minimizer of the constrained variational problem for \(E(u_1, u_2, \cdots, u_m)\), then the standing wave is called the ground-state solution of (1.2). It is our aim in this manuscript to prove existence and stability results for ground-state solutions of (1.2). A solution of (1.2) is called a vector solution if none of its component vanishes.

2. Statement of Results

In this manuscript, we will employ the technique used in [18] to show the existence and stability of ground-state solutions to (1.2). The crucial ideas are the following.

1. The constraints on the \(L^2\)-norms of the components are independently prescribed, in contrast to the work in [13, 17] where the constraints are related to one another.

2. Strict sub-additivity of the function \(I\) as defined below must be verified.

As usual, strict sub-additivity is difficult to verify. Indeed, even though the main idea here is similar to the one used in [13, 17], it is at item 2 where the technique departs from the others.

Throughout the present paper, the real symmetric matrix of coefficients \(B = (b_{ij})\) is assumed to satisfy
\[
b_{ij} > 0, \tag{2.1}
\]
for all \(i, j = 1, \cdots, m\). For all \(\mathcal{U} = (u_1, \cdots, u_m) \in X_m\), let
\[
E(\mathcal{U}) = \int_{\mathbb{R}} \sum_{j=1}^m \left( \frac{d}{dx} u_j \right)^2 dx - \frac{1}{p} \sum_{i,j=1}^m b_{ij} |u_i|^p |u_j|^p dx.
\]
For all \(S = (s_1, s_2, \cdots, s_m)\) with \(s_j \geq 0, j = 1, \cdots, m\), define the real number \(I\) as follows:
\[
I(S) = \inf \{ E(\mathcal{U}) : \mathcal{U} = (u_1, u_2, \cdots, u_m) \in X_m, \|u_j\|_2^2 = s_j, j = 1, 2, \cdots, m \} \tag{2.2}
\]
and the set of minimizers for \(I(S)\) as
\[
G(S) = \{ \mathcal{U} = (u_1, u_2, \cdots, u_m) \in X_m : E(\mathcal{U}) = I(S), \|u_j\|_2^2 = s_j, j = 1, 2, \cdots, m \}.
\]
**Theorem 2.1.** Let $2 \leq p < 3$ and suppose that the real symmetric matrix $B$ satisfies (2.1). Then the following statements are true for all $\Lambda = (\lambda_1, \ldots, \lambda_m)$ such that $\lambda_j > 0, j = 1, \ldots, m$.

1. Every minimizing sequence $\{(u_1^{(n)}, \ldots, u_m^{(n)}) \in X_m\}$ is relatively compact in $X_m$ up to translation. In particular, the set $G(\Lambda)$ is non-empty.
2. Each function $(\phi_1, \phi_2, \ldots, \phi_m) \in G(\Lambda)$ is a solution of (1.5) for some $w_j > 0, j = 1, \ldots, m$ and therefore when substituted into (1.3) yields a (standing-wave) solitary-wave solution of (1.2).
3. For every $(u_1, \ldots, u_m) \in G(\Lambda)$, there exist numbers $x_0, \theta_j \in \mathbb{R}, j = 1, \ldots, m$ and even functions $\phi_j(x) > 0, j = 1, \ldots, m$, $\forall x \in \mathbb{R}$ such that $u_j(x) = e^{i\theta_j} \phi_j(x + x_0), j = 1, \ldots, m$. Moreover, the functions $\phi_j, j = 1, \ldots, m$ are infinitely differentiable and exponentially decreasing.
4. For every $\epsilon > 0$ given, there exists a $\delta > 0$ such that if
   \[
   \inf_{(\Phi_1, \ldots, \Phi_m) \in G} \sum_{j=1}^m \|u_j - \Phi_j\|_{H^k_1(\mathbb{R})} < \delta,
   \]
   then the solution $(u_1(x, t), \ldots, u_m(x, t))$ with $(u_1(x, 0), \ldots, u_m(x, 0)) = (u_1, \ldots, u_m)$ satisfies
   \[
   \inf_{(\Phi_1, \ldots, \Phi_m) \in G} \sum_{j=1}^m \|u_j(x, t) - \Phi_j\|_{H^k_1(\mathbb{R})} < \epsilon
   \]
   for all $t \in \mathbb{R}$.

Some remarks are needed in order to understand the Theorem.

**Remark 2.1.** For $m = 2, p = 2$ the above Theorem has been proved in [18], and for $m = 3$ the above Theorem recovers Theorem 3.6 in [4]. Therefore, the result obtained in the present paper generalizes that of [4, 18].

**Remark 2.2.** When the constraints are not independently chosen, such as when
\[
\begin{align*}
b_{ii} &= a, & b_{ij} &= b > 0, & i \neq j, & a + (m-1)b > 0, & \lambda_i &= \frac{1}{a + (m-1)b} & \quad \text{(2.3)}
\end{align*}
\]
for all $i, j = 1, 2, \ldots, m$, it was proved in [16] that the set of minimizers $G(\Lambda)$ consists of, up to translations, vector solutions with each component being multiple of the hyperbolic function $\text{sech}$. (Notice that in the condition (2.3), the constant $a$ is allowed to be negative as well, provided that $b > 0$ is large enough.)

**Remark 2.3.** For $m = 2, 2 \leq p$, a minimizer of the variational problem (2.2) was shown to exist by a mountain pass type argument in [2].

### 3. Variational Problem

The following two Lemmas are straightforward to see hence proofs are omitted.

**Lemma 3.1.** Suppose $2 \leq p < 3$. Let $M > 0$ be a given constant. Suppose that $\|f_j\|_{H^k_1(\mathbb{R})} \leq M$ and $\|g_j\|_{H^k_1(\mathbb{R})} \leq M$, $1 \leq j \leq m$, then the following inequality
\[
|E(f_1, \ldots, f_m) - E(g_1, \ldots, g_m)| \leq C(M) \sum_{j=1}^m \|f_j - g_j\|_{H^k_1(\mathbb{R})}
\]
holds where the constant $C(M)$ is dependent only on $M$.

**Lemma 3.2.** Let $2 \leq p < 3$. Suppose that $\{(f_1^{(n)}, \ldots, f_m^{(n)})\}$ is a uniformly bounded sequence in $X_m$ such that $\|f_j^{(n)}\|_2^2 \to \theta_j$ as $n \to \infty$ for all $j = 1, \ldots, m$. Then, for arbitrary $\epsilon > 0$ there exists an $N > 0$ such that
\[
E(f_1^{(n)}, \ldots, f_m^{(n)}) \geq I(\theta_1, \ldots, \theta_m) - \epsilon,
\]
for all $n > N$.

**Lemma 3.3.** Let $2 \leq p < 3$. For all $S = (s_1, \ldots, s_m), s_i \geq 0, i = 1, 2, \ldots, m$, one has
(1) \[ I(\mathcal{S}) > -\infty. \]

(2) If there exists some \( k \) such that \( s_k > 0 \), then
\[ I(\mathcal{S}) < 0. \]

Proof. To see \( I(\mathcal{S}) > -\infty \), notice that by Gagliardo-Nirenberg inequality, \( \forall u \in H^1_c(\mathbb{R}) \) one has
\[ \|u\|_q^q \leq C \|u\|_2^{q+1} \|u_x\|_2^{q-1}, \] (3.1)
where \( 2 < q < \infty \). Let \( U = (u_1, \cdots, u_m) \in X_m \) with \( \|u_i\|_2^2 = s_i \) then by definition of the energy function and the inequality (3.1)
\[ E(U) \geq \sum_{j=1}^{m} \int_{\mathbb{R}} \left| \frac{du_j}{dx} \right|^2 dx - C \sum_{j=1}^{m} |u_j|^{2p} dx \]
\[ \geq \sum_{j=1}^{m} \int_{\mathbb{R}} \left| \frac{du_j}{dx} \right|^2 dx - C \sum_{j=1}^{m} \left( \int_{\mathbb{R}} \left| \frac{du_j}{dx} \right|^2 dx \right)^{\frac{p}{2}} \]
\[ > -\infty \]
where the conditions \( \|u_i\|_2^2 = s_i \), \( 1 \leq i \leq m \) have been used in the second inequality and \( C \) denotes various constants whose precise values are not of importance.

To see that \( I(\mathcal{S}) < 0 \), take a function \( f \in H^1_c(\mathbb{R}) \) such that \( \|f\|_2 = \|f_x\|_2 = 1 \) and let \( f_r(x) = \sqrt{r}f(rx) \) for any number \( r > 0 \). Then \( \|f_r\|_2 = 1 \), \( \|f_r\|_{2p}^2 = r^{p-1}\|f\|_{2p}^2 \), \( \|f'_r\|_2^2 = r^2 \). Therefore, for \( U_r = (\sqrt{s_1}f_r, \cdots, \sqrt{s_m}f_r) \), one has
\[ E(U_r) = r^2 \sum_{j=1}^{m} s_j - \frac{r^{p-1}}{p} \left( \sum_{i,j=1}^{m} b_{ij}s_i s_j \right) \|f\|_{2p}^2. \]

By the assumption that there exists some \( s_k > 0 \), we deduce that
\[ E(U_r) \leq r^2 \sum_{j=1}^{m} s_j - \frac{r^{p-1}}{p} b_{kk}s_k^p \|f\|_{2p}^2 \]
where the assumption (2.1) has been used. By taking \( r = r_0 \) small enough, we have \( E(U_{r_0}) < 0 \) and consequently, \( I(\mathcal{S}) < 0 \). \( \square \)

The following Lemma guarantees that minimizing sequences must be bounded uniformly in \( H^1_c(\mathbb{R}) \) and the \( L^{2p} \)-norm bounded away from zero for all large \( n \).

**Lemma 3.4.** Let \( 2 \leq p < 3 \) and \( S = (s_1, \cdots, s_m) \), \( s_i \geq 0 \), \( i = 1, 2, \cdots, m \). If \( \{(u_1^{(n)}, \cdots, u_m^{(n)})\} \) is any minimizing sequence for \( I(\mathcal{S}) \), then
(1) \( \sum_{j=1}^{m} \|u_j^{(n)}\|_{H^1_c(\mathbb{R})}^2 \) is a uniformly bounded sequence;
(2) If \( s_k > 0 \), then there exist \( \delta = \delta(k) > 0 \), \( \eta = \eta(k) > 0 \) and \( \kappa > 0 \) such that
\[ \sum_{j=1}^{m} \|u_j^{(n)}\|_{2p}^2 \geq \kappa, \] (3.2)
\[ \frac{1}{p} b_{kk} \|u_k^{(n)}\|_{2p}^2 - \frac{2}{p} \sum_{j \neq k}^{m} b_{jk} \|u_j^{(n)} u_k^{(n)}\|_p^p \leq -\delta(k) < 0, \] (3.3)
and for all sufficiently large \( n \),
\[ \|\frac{du_k^{(n)}}{dx}\|_2^2 \geq \eta(k). \] (3.4)
Proof. Let $U_n = (u_n^{(1)}, \ldots, u_n^{(m)})$ be any minimizing sequence for $I(S)$, then by Gagliardo-Nirenberg inequality,

$$
\sum_{j=1}^{m} \frac{\|du_j^{(n)}\|_2^2}{dx} = E(U_n) + \frac{1}{p} \sum_{i,j=1}^{m} \int b_{ij} |u_i^{(n)} - u_j^{(n)}|^p dx \leq E(U_n) + C \sum_{j=1}^{m} \frac{\|du_j^{(n)}\|_2^2}{dx} < \infty.
$$

Thus, $\sum_{j=1}^{m} \|u_j^{(n)}\|_{H^2_0(\mathbb{R})}^2$ is a uniformly bounded sequence.

To prove that the estimate (3.2) holds, assume to the contrary that

$$
\sum_{j=1}^{m} \|u_j^{(n)}\|_{2p}^2 \to 0, \quad \text{as } n \to \infty.
$$

Using the inequality

$$
\int_{\mathbb{R}} |f|^p dx \leq \left( \int_{\mathbb{R}} |f|^{2p} dx \right)^{1/2} \left( \int_{\mathbb{R}} |g|^{2p} dx \right)^{1/2},
$$

one has

$$
I(S) = \lim_{n \to \infty} \sum_{j=1}^{m} \frac{\|du_j^{(n)}\|_2^2}{dx} \geq 0,
$$

which contradicts the Lemma 3.3. Consequently, the estimate (3.2) must hold true.

Suppose that the estimate (3.3) is false, then by passing to a subsequence if necessary, we may assume there exists a minimizing sequence such that

$$
\liminf_{n \to \infty} \left[ \frac{\|du_k^{(n)}\|_2^2}{dx} - \frac{1}{p} \|b_{kk}\|_{2p} \|u_k^{(n)}\|_{2p}^p - \frac{2}{p} \sum_{j \neq k} b_{jk} \|u_j^{(n)}\|_2^2 \right] \geq 0.
$$

Thus,

$$
I(S) \geq \liminf_{n \to \infty} \left[ \sum_{j \neq k} \frac{\|du_j^{(n)}\|_2^2}{dx} - \frac{1}{p} \sum_{i,j \neq k} b_{ij} \|u_i^{(n)}\|_2^2 \right].
$$

Now, pick a function $\phi \in H^1_0(\mathbb{R})$ such that $\|\phi\|_2 = \|dx\phi\|_2 = 1$ and let $\phi_r(x) = \sqrt{r}\phi(rx)$ for some $r > 0$. For all $n$,

$$
I(S) \leq E(u_1^{(n)}, \ldots, u_{k-1}^{(n)}, s_k \phi_r, u_{k+1}^{(n)}, \ldots, u_m^{(n)}).
$$

On the other hand, if we define

$$
\gamma = s_k r^2 - \frac{p-1}{p} b_{kk} s_k^p \|\phi\|_2^2 - \frac{2}{p} \sum_{j \neq k} b_{jk} \|\phi_r u_j^{(n)}\|_p^p,
$$

then for sufficiently small $r > 0$, we must have $\gamma < 0$ because $b_{kk} > 0$, $s_k > 0$, $2 \leq p < 3$. Consequently, for all $n$,

$$
I(S) \leq E(u_1^{(n)}, \ldots, u_{k-1}^{(n)}, \sqrt{s_k} \phi_r, u_{k+1}^{(n)}, \ldots, u_m^{(n)})
$$

$$
= \left[ \sum_{j \neq k} \frac{\|du_j^{(n)}\|_2^2}{dx} - \frac{1}{p} \sum_{i,j \neq k} b_{ij} \|u_i^{(n)}\|_2^2 \right] + \gamma.
$$

But then this leads to

$$
I(S) \leq I(S) + \gamma,
$$

which is a contradiction to the previously established statement that $\gamma < 0$.

Finally, the inequality (3.4) follows immediately from (3.3), the Gagliardo-Nirenberg inequality and part (1) of this Lemma.

Following approach used in [1], we show in the next two Lemmas that the value of $E(u_1, \ldots, u_m)$ decreases when $u_1, \ldots, u_m$ are replaced by $|u_1|, \ldots, |u_m|$ and when $|u_1|, |u_2|, \ldots, |u_m|$ are symmetrically rearranged. It is straightforward to see the next Lemma, using the fact that

$$
\int_{\mathbb{R}} |f|^2 dx \leq \int_{\mathbb{R}} |f_x|^2 dx.
$$
Lemma 3.5. Let $2 \leq p < 3$. For all $(u_1, \ldots, u_m) \in X_m$, one has
\[ E(|u_1|, \ldots, |u_m|) \leq E(u_1, \ldots, u_m). \]

We recall here (see also [1, 10]) the definition of symmetric decreasing rearrangement of a function. Let $w : \mathbb{R} \to [0, \infty)$ be a non-negative function. If $\{x : w(x) > y\}$ has finite measure $m(w, y)$ for all $y > 0$, then the symmetric decreasing rearrangement $w^*$ of $w$ is defined by
\[ w^*(x) = \inf\{y > 0 : \frac{1}{2}m(w, y) \leq x\}. \]

Notice that if $(u_1, \ldots, u_m) \in X_m$ then $(|u_1|, \ldots, |u_m|) \in X_m$.

Lemma 3.6. Let $2 \leq p < 3$. For all $(u_1, \ldots, u_m) \in X_m$, it must be true that
\[ E(|u_1|^*, \ldots, |u_m|^*) \leq E(u_1, \ldots, u_m). \]

Proof. Using the following important facts (for the proofs of those, see [10]):
\begin{align*}
  &a) \int_{\mathbb{R}} (|f|)^{2p} \, dx = \int_{\mathbb{R}} |f|^{2p} \, dx; \\
  &b) \int_{\mathbb{R}} (|f|)^p (|g|^*)^p \, dx \geq \int_{\mathbb{R}} |f|^p |g|^p \, dx; \\
  &c) \int_{\mathbb{R}} (|f|^*)^2 \, dx \leq \int_{\mathbb{R}} |f|^2 \, dx;
\end{align*}
and $b_{ij} > 0$, $i \neq j$, the Lemma follows immediately. \hfill \square

The next Lemma is crucial in obtaining the strict sub-additivity of the function $I(\Lambda)$ needed in ruling out dichotomy of minimizing sequences. We refer readers to [1] for the proof of this.

Lemma 3.7. Suppose $u$ and $v$ are non-negative, even, $C^\infty$-functions with compact supports in $\mathbb{R}$, which are non-increasing on $\{x : x \geq 0\}$. Let $x_1$ and $x_2$ be numbers such that $u(x_1 + x_1)$ and $v(x_1 + x_2)$ have disjoint supports, and define
\[ w(x) = u(x_1 + x_1) + v(x_1 + x_2). \]
Let $w^* : \mathbb{R} \to \mathbb{R}$ be the symmetric decreasing rearrangement of $w$. Then the distributional derivative $(w^*)'$ is in $L^2$ and satisfies
\[ \| (w^*)' \|_2^2 \leq \| w' \|_2^2 - \frac{3}{4} \min\{ \| u' \|_2^2, \| v' \|_2^2 \}. \]

Lemma 3.8. Let $2 \leq p < 3$. Suppose that for each $N < m$, there exists a positive minimizer for the $N$-coupled variational problem. For $\Lambda = (\lambda_1, \ldots, \lambda_m)$, $\lambda_j > 0$ and arbitrary $\Theta = (\theta_1, \ldots, \theta_m)$ with $0 \leq \theta_j \leq \lambda_j$ for $1 \leq j \leq m$ and $0 < \theta_1 + \cdots + \theta_m < \lambda_1 + \cdots + \lambda_m$, the following inequality
\[ I(\Lambda) < I(\Theta) + I(\Lambda - \Theta). \]
holds true.

Proof. Following closely the argument used in [1], we claim that for $j = 1, 2, \ldots, m$, one can choose minimizing sequences $(u_1^{(n)}, \ldots, u_m^{(n)})$ for $I(\Theta)$ and $(v_1^{(n)}, \ldots, v_m^{(n)})$ for $I(\Lambda - \Theta)$ such that for all $n \in \mathbb{N}$, $u_1^{(n)}, \ldots, u_m^{(n)}$ and $v_1^{(n)}, \ldots, v_m^{(n)}$:
\begin{enumerate}
  \item[i)] are real-valued and non-negative on $\mathbb{R}$;
  \item[ii)] belong to $H^1_\infty(\mathbb{R})$ and have compact supports;
  \item[iii)] are even functions;
  \item[iv)] are non-increasing functions of $x$, for all $x \geq 0$;
  \item[v)] are $C^\infty$-functions; and
  \item[vi)] $\| u_j^{(n)} \|_2^2 = \lambda_j$, $\| v_j^{(n)} \|_2^2 = \lambda_j - \theta_j$, for $j = 1, \ldots, m$.
\end{enumerate}
Without loss of generality, we can take $j = 1$ as the cases $1 < j \leq m$ are exactly the same. Moreover, we can assume that $0 < \theta_j < \lambda_j$, as otherwise just simply take $u_j^{(n)} = 0$ for $\theta_j = 0$ or $v_j^{(n)} = 0$ for $\theta_j = \lambda_j$. Now, let $(u_1^{(n)}, \ldots, u_m^{(n)})$ be any minimizing sequence for $I(\Theta)$. Since smooth functions with compact supports are dense in $H^1_\infty(\mathbb{R})$ and $E : X_m \to \mathbb{R}$ is continuous (see Lemma 3.1), we can approximate $(u_1^{(n)}, \ldots, u_m^{(n)})$ by functions $(w_1^{(n)}, \ldots, w_m^{(n)})$ which have compact supports and which still form a minimizing sequence for $I(\Theta)$. Then by Lemma 3.6,
\[ (z_1^{(n)}, \ldots, z_m^{(n)}) := (|w_1^{(n)}|^*, \ldots, |w_m^{(n)}|^*) \]
is still a minimizing sequence for $I(\Theta)$, and for each $n \in \mathbb{N}$, $z_1^{(n)}, \ldots, z_m^{(n)}$ satisfy $i) - iv)$. 

Notice next that if \( \phi \) is any non-negative, even, \( C^\infty \), decreasing function for \( x \geq 0 \) with compact support, then the convolution of \( \phi \) with any function \( f \) satisfying properties \( i) - iv) \)

\[
f * \phi(x) = \int_{\mathbb{R}} f(x-y)\phi(y)dy
\]
also satisfies \( i) - iv). Using “approximation to the identity”

\[
\phi_\epsilon(x) = \frac{1}{\epsilon} \phi(\frac{x}{\epsilon}), \quad \text{for } \epsilon > x \quad \text{with} \quad \int_{\mathbb{R}} \phi(x)dx = 1,
\]
then \( f * \phi_\epsilon \to f \) as \( \epsilon \to 0 \). Thus, by choosing \( \phi \) satisfying \( \int_{\mathbb{R}} \phi(x)dx = 1 \) to be any non-negative, even, \( C^\infty \), decreasing function for \( x \geq 0 \) with compact support, and defining

\[
(h_1^{(n)}, \ldots, h_m^{(n)}) = (z_1^{(n)} * \phi_{\epsilon_n}, \ldots, z_m^{(n)} * \phi_{\epsilon_n})
\]
with \( \epsilon_n \) arbitrarily small for \( n \) large, then \( (h_1^{(n)}, \ldots, h_m^{(n)}) \) satisfy \( i) - v) \).

Set

\[
\nu_j^{(n)} = \sqrt{\| h_j^{(n)} \|_2}, \quad j = 1, \ldots, m,
\]
(which is possible as \( \| h_j^{(n)} \|_2 > 0 \) for \( n \) large when \( \theta_j > 0 \)), then \( (u_1^{(n)}, \ldots, u_m^{(n)}) \) satisfies all \( i) - vi) \).

For each \( n \), choose a number \( x_n \) such that \( u_j^{(n)}(x_n) \) and \( v_j^{(n)}(x) := v_j^{(n)}(x + x_n) \) have disjoint supports for all \( j \in J := \{ j \in [1, m] : 0 < \theta_j < \lambda_j \} \). Define

\[
f_j^{(n)} = (u_j^{(n)} + v_j^{(n)})^*, \quad j \in J,
\]
and

\[
f_j^{(n)} = u_j^{(n)} + v_j^{(n)}, \quad j \notin J.
\]
Then \( \| f_j^{(n)} \|_2^2 = \lambda_j, j = 1, \ldots, m; \) so

\[
I(\Lambda) \leq E(f_1^{(n)}, \ldots, f_m^{(n)}).
\]

On the other hand, Lemma 3.7 guarantees that

\[
\sum_{j=1}^m \| \frac{df_j^{(n)}}{dx} \|_2^2 = \sum_{j \in J} \| \frac{df_j^{(n)}}{dx} \|_2^2 + \sum_{j \notin J} \| \frac{df_j^{(n)}}{dx} \|_2^2
\]
\[
\leq \sum_{j \in J} \| \frac{du_j^{(n)}}{dx} + \frac{dv_j^{(n)}}{dx} \|_2^2 - K_n + \sum_{j \notin J} \| \frac{du_j^{(n)}}{dx} + \frac{dv_j^{(n)}}{dx} \|_2^2
\]
\[
= \sum_{j \in J} \left( \| \frac{du_j^{(n)}}{dx} \|_2^2 + \| \frac{dv_j^{(n)}}{dx} \|_2^2 \right) - K_n + \sum_{j \notin J} \left( \| \frac{du_j^{(n)}}{dx} \|_2^2 + \| \frac{dv_j^{(n)}}{dx} \|_2^2 \right)
\]
\[
= \sum_{j=1}^m \left( \| \frac{du_j^{(n)}}{dx} \|_2^2 + \| \frac{dv_j^{(n)}}{dx} \|_2^2 \right) - K_n
\]

where

\[
K_n = \frac{3}{4} \sum_{j \in J} \min \{ \| \frac{du_j^{(n)}}{dx} \|_2^2, \| \frac{dv_j^{(n)}}{dx} \|_2^2 \},
\]

and \( K_n = 0 \) when \( J \) is an empty set. Moreover, from properties of rearrangement (3.5) we have

\[
\| f_j^{(n)} \|_{2p}^2 = \| u_j^{(n)} \|_{2p}^2 + \| v_j^{(n)} \|_{2p}^2,
\]

\[
\| f_j^{(n)} \|_p \geq \| u_j^{(n)} \|_p + \| v_j^{(n)} \|_p.
\]

Thus, (3.7) and (3.8) combine to give for all \( n \),

\[
I(\Lambda) \leq E(f_1^{(n)}, \ldots, f_m^{(n)}) \leq E(u_1^{(n)}, \ldots, u_m^{(n)}) + E(v_1^{(n)}, \ldots, v_m^{(n)}) - K_n.
\]

Hence

\[
I(\Lambda) \leq I(\Theta) + I(\Lambda - \Theta) - \lim_{n \to \infty} K_n.
\]
Now, we consider separately the following two cases.

**Case 1:** $J$ is a non-empty set. By \((3.4)\) in Lemma 3.4, one has

$$K_n \geq \eta := \frac{3}{4} \min \{\eta(k) : k \in J \} > 0$$

for all $n$ large. From \((3.9)\), we then have

$$I(\Lambda) \leq I(\Theta) + I(\Lambda - \Theta) - \eta < I(\Theta) + I(\Lambda - \Theta).$$

**Case 2:** $J$ is an empty set, namely, for all $j$, either $\theta_j = 0$ or $\theta_j = \lambda_j$. Without loss of generality, we can assume that there exists $1 < N < m$ such that

$$\theta_1 = \cdots = \theta_N = 0, \quad \theta_j = \lambda_j, j > N.$$

By the assumption, we can take a minimizer $(0, \cdots, 0, \psi_{N+1}, \cdots, \psi_m) \in X_m$ of $I(\Theta)$ and a minimizer $(\psi_1, \cdots, \psi_N, 0, \cdots, 0) \in X_m$ of $I(\Lambda - \Theta)$ such that $\psi_j$ are all positive functions. Thus,

$$I(\Lambda) \leq E(\psi_1, \cdots, \psi_m)$$

$$= E(0, \cdots, 0, \psi_{N+1}, \cdots, \psi_m) + E(\psi_1, \cdots, \psi_N, 0, \cdots, 0) - \frac{2}{p} \sum_{i \leq N, j > N} b_{ij} \|\psi_i^{(n)}\|_p \|\psi_j^{(n)}\|_p$$

$$= I(\Theta) + I(\Lambda - \Theta) - \frac{2}{p} \sum_{i \leq N, j > N} b_{ij} \|\psi_i^{(n)}\|_p \|\psi_j^{(n)}\|_p$$

$$< I(\Theta) + I(\Lambda - \Theta)$$

where $b_{ij} > 0$ and the fact that $\psi_j$ are all positive functions have been used in the last inequality.

Thus, the Lemma is proved. \hfill \Box

Let $\{(u_1^{(n)}, \cdots, u_m^{(n)})\} \in X_m$ be a minimizing sequence for $E$ and consider a sequence of nondecreasing functions

$$M_n : [0, \infty) \mapsto [0, \sum_{j=1}^m \lambda_j]$$

as follows

$$M_n(s) = \sup_{y \in \mathbb{R}} \int_{y-s}^{y+s} \sum_{j=1}^m |u_j^{(n)}(x)|^2 \, dx.$$

As $M_n(s)$ is a uniformly bounded sequence of nondecreasing functions in $s$, one can show that it has a subsequence, which is still denoted as $M_n$, that converges point-wisely to a nondecreasing limit function

$$M(s) : [0, \infty) \mapsto [0, \sum_{j=1}^m \lambda_j].$$

Let

$$\rho = \lim_{s \to \infty} M(s) := \lim_{s \to \infty} \lim_{n \to \infty} M_n(s) = \lim_{s \to \infty} \lim_{n \to \infty} \sup_{y \in \mathbb{R}} \int_{y-s}^{y+s} \sum_{j=1}^m |u_j^{(n)}(x)|^2 \, dx.$$

Then $0 \leq \rho \leq \sum_{j=1}^m \lambda_j$.

Lions’ Concentration Compactness Lemma \([11, 12]\) shows that there are three possibilities for the value of $\rho$:

(i) Case 1: (Vanishing) $\rho = 0$. Since $M(s)$ is non-negative and nondecreasing, this is equivalent to saying

$$M(s) = \lim_{n \to \infty} \sup_{y \in \mathbb{R}} \int_{y-s}^{y+s} \sum_{j=1}^m |u_j^{(n)}(x)|^2 \, dx = 0,$$

for all $s < \infty$, or

(ii) Case 2: (Dichotomy) $\rho \in (0, \sum_{j=1}^m \lambda_j)$, or
(iii) Case 3: (Compactness) \( \rho = \sum_{j=1}^{m} \lambda_j \), which implies that there exists \( \{y_n\} \in \mathbb{R} \) such that
\[
\sum_{j=1}^{m} |u_j^{(n)}(x + y_n)|^2 \leq \text{tight, namely, for all } \epsilon > 0, \text{ there exists } s < \infty \text{ such that}
\]
\[
\int_{y_n-s}^{y_n+s} |u_j^{(n)}(x)|^2 dx \geq \sum_{j=1}^{m} \lambda_j - \epsilon.
\]
(3.11)
for all sufficiently large \( n \).

The following Lemma is well-known (See, for example, [1, 10]).

Lemma 3.9. Suppose \( f_n \) is a sequence of functions which is bounded in \( H^1_0(\mathbb{R}) \) and which satisfies
for some \( r > 0 \),
\[
\limsup_{n \to \infty} \frac{1}{r} \int_{y-r}^{y+r} |f_n|^2 dx = 0.
\]
Then, \( \lim_{n \to \infty} \|f_n\|_q = 0 \) for any \( q > 2 \).

The next Lemma says that vanishing of minimizing sequences cannot occur.

Lemma 3.10. Let \( 2 \leq p < 3 \). For any minimizing sequence, \( \rho > 0 \).

Proof. Suppose to the contrary that \( \rho = 0 \). Then (3.12) holds for all \( w_n = u_j^{(n)}, j = 1, \ldots, m \).
Thus, Lemma 3.9 says that for all \( q > 2, u_j^{(n)} \to 0 \) in \( L^q \)-norm. In particular, for all \( j = 1, \ldots, m \),
\[
\|u_j^{(n)}\|_{2p} \to 0.
\]
By Hölder’s inequality,
\[
\|u_j^{(n)} u_j^{(n)}\|_p \leq \|u_j^{(n)}\|_{2p} \|u_j^{(n)}\|_{2p} \to 0.
\]
Hence
\[
I(\lambda) = \lim_{n \to \infty} \int \sum_{j=1}^{m} \left( \frac{du_j^{(n)}(x)}{dx} \right)^2 dx \geq 0,
\]
which contradicts Lemma 3.3. \( \square \)

We now choose a function \( \Gamma \in C^\infty_0([-2, 2]) \) such that \( \Gamma \equiv 1 \) on \([-1, 1]\), and let \( \Pi \in C^\infty(\mathbb{R}) \) be such that \( \Gamma^2 + \Pi^2 \equiv 1 \) on \( \mathbb{R} \). For each \( r > 0 \), define \( \Gamma_r(x) = \Gamma(x/r) \) and \( \Pi_r(x) = \Pi(x/r) \). Let \( \epsilon > 0 \) be given; for all sufficiently large \( r \) we have
\[
\rho - \epsilon < M(r) \leq M(2r) \leq \rho.
\]
Assume for the moment that such a value of \( r \) has been chosen. Then one can choose \( N \) so large that
\[
\rho - \epsilon < M_n(r) \leq M_n(2r) < \rho + \epsilon
\]
for all \( n \geq N \). Consequently, for each \( n \geq N \), one can find \( y_n \) such that
\[
\int_{y_n-r}^{y_n+r} \sum_{j=1}^{m} |u_j^{(n)}(x)|^2 dx \geq \rho - \epsilon.
\]
Certainly, one has
\[
\int_{y_n-2r}^{y_n+2r} \sum_{j=1}^{m} |u_j^{(n)}(x)|^2 dx < \rho + \epsilon.
\]
For all \( j = 1, \ldots, m \), define
\[
u_j^{(n)}(x) = \Gamma_r(x - y_n) u_j^{(n)}(x), \quad \nu_j^{(n)}(x) = \Pi_r(x - y_n) u_j^{(n)}(x).
\]
The next Lemma describes the behavior of minimizing sequences in the case \( 0 < \rho < \sum_{j=1}^{m} \lambda_j \).
The proof of this Lemma is similar to that in [17] and hence is omitted.

Lemma 3.11. Let \( 2 \leq p < 3 \). For every \( \epsilon > 0 \) given, there exists an \( N > 0 \) such that for every \( n \geq N \),
\[
|\sum_{j=1}^{m} \|u_j^{(n)}\|_2^2 - \rho| < \epsilon;
\]
1.
2. \[ E(u^{(n)}_1, \ldots, u^{(n)}_m) \geq E(u^{(n)}_{1,1}, \ldots, u^{(n)}_{m,1}) + E(u^{(n)}_{1,2}, \ldots, u^{(n)}_{m,2}) - C\epsilon, \]

for some constant \( C > 0 \) independent of \( n \).

The dichotomy of minimizing sequences can now be prevented by the next Proposition.

**Proposition 3.1.** Let \( 2 \leq p < 3 \). For every minimizing sequence, \( \rho = 0 \) or \( \rho = \sum_{j=1}^{m} \lambda_j \).

**Proof.** Assume to the contrary that \( 0 < \rho < \sum_{j=1}^{m} \lambda_j \). By Lemma 3.11, we can assume that there exists a subsequence \( \{(u^{(n)}_{1,1}, \ldots, u^{(n)}_{m,1}) \} \) of \( \{(u^{(n)}_1, \ldots, u^{(n)}_m) \} \) such that

\[ \|u^{(n)}_{j,1}\|^2_2 \rightarrow \theta_j, \quad \text{as} \quad k \rightarrow \infty \]

for \( j = 1, \ldots, m \). Then, \( \theta_j \in [0, \lambda_j] \) and \( \sum_{j=1}^{m} \theta_j = \rho \).

By Lemma 3.2, for arbitrary \( \epsilon > 0 \) given, there exists a \( K \) such that

\[ E(u^{(n)}_{1,1}, \ldots, u^{(n)}_{m,1}) \geq I(\theta_1, \ldots, \theta_m) - \epsilon \quad (3.13) \]

for all \( k > K \). Notice that

\[ \|u^{(n)}_{j,2}\|^2_2 = \lambda_j - \|u^{(n)}_{j,1}\|^2_2 \rightarrow \lambda_j - \theta_j, \]

as \( k \rightarrow \infty \). Similarly, one has

\[ E(u^{(n)}_{1,2}, \ldots, u^{(n)}_{m,2}) \geq I(\lambda_1 - \theta_1, \ldots, \lambda_m - \theta_m) - \epsilon \quad (3.14) \]

for all sufficiently large \( k \). Combining (3.13) and (3.14), Lemma 3.11 implies

\[ I(\lambda_1, \ldots, \lambda_m) \geq I(\theta_1, \ldots, \theta_m) + I(\lambda_1 - \theta_1, \ldots, \lambda_m - \theta_m), \]

which contradicts Lemma 3.8 as \( 0 < \sum_{j=1}^{m} \theta_j < \sum_{j=1}^{m} \lambda_j \).

**Remark 3.1.** By Lemma 3.10 and the above Proposition, we conclude that \( \rho = \sum_{j=1}^{m} \lambda_j \). This implies that \( \theta_j = \lambda_j \) for all \( j = 1, \ldots, m \). Consequently, we can rule out the vanishing of the minimizing sequence in the true sense; that is, for all \( s > 0 \),

\[ \lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}} \int_{y-s}^{y+s} |u^{(n)}_j(x)|^2 dx > 0 \]

for all \( j = 1, \ldots, m \).

With all the above calculations at hand, we now proceed to prove Theorem 2.1.

**Proof.** (of Theorem 2.1) As we have ruled out both the vanishing and dichotomy cases, it follows from Lions’ Compactness Lemma [11, 12] that every minimizing sequence must be compact up to translations. However, Lemma 3.8 is proved under an additional condition that there exists a positive minimizer for \( N \)-coupled variational problem for each \( N < m \). So, we need to prove that the condition (i.e. item (3) in the present Theorem) is true by an inductive argument. To begin with, by the Lemma 2.3 in [4], item (3) holds for the case \( m=2 \). Next, suppose that item (3) in Theorem 2.1 is true for all cases \( m < k \). Let us show that it also holds for the case \( m = k + 1 \).

In fact, by the inductive hypothesis, for any \( N < k + 1 \) there exists a positive minimizer for the \( N \)-coupled variational problem, therefore the statement (1) in Theorem 2.1 for \( m = k + 1 \) follows from Lemma 3.8 again.

To see the validity of statements (2) and (3), notice that the Lagrange multiplier principle guarantees that there are real numbers \( w_j, j = 1, \ldots, m \) such that

\[ -(u_i)_{xx} + w_i u_i = \sum_{j=1}^{m} b_{ij} |u_j|^p |u_i|^{p-2} u_i, \quad i = 1, \ldots, m \quad (3.15) \]

holds at least in the sense of distributions. A straightforward bootstrapping argument reveals that indeed (3.15) holds true in the classical sense as well. Multiplying the equation (3.15) by \( u_i \) for \( i = 1, \ldots, m \) and integrating over \( \mathbb{R} \), we obtain

\[ \| (u_i)_{x} \|_2^2 - \sum_{j=1}^{m} b_{ij} \| u_i u_j \|_p^p = -w_i \lambda_i, \quad i = 1, \ldots, m \quad (3.16) \]
But (3.3) in Lemma 3.4 implies that
\[
\|(u_j)_x\|^2_2 - \sum_{j=1}^{m} b_{ij} \|u_i u_j\|^p_p \\
= \left[\|(u_j)_x\|^2_2 - \frac{1}{p} b_{ii} \|u_i\|^{2p}_p - \frac{2}{p} \sum_{j \neq i}^m b_{ij} \|u_i u_j\|^p_p\right] \\
- \frac{p-1}{p} b_{ii} \|u_i\|^{2p}_p - \frac{2}{p} \sum_{j \neq i}^m b_{ij} \|u_i u_j\|^p_p < 0,
\]
for \(i = 1, \ldots, m\) because of the assumption (2.1) and \(p \geq 2\). Consequently, (3.16) and (3.17) assert that \(w_j > 0\) for \(j = 1, \ldots, m\).

It is left to prove statement (3). Set \(\phi_j(x) = |u_j(x)|\). By Lemma 3.5, \((\phi_1, \ldots, \phi_m)\) is also a minimizer of the variational problem, hence (3.15) is satisfied by \((\phi_1, \ldots, \phi_m)\). The Lagrange multipliers stay the same as they are determined by (3.16), which are unchanged when \((u_1, \ldots, u_m)\) is replaced by \((\phi_1, \ldots, \phi_m)\). We can rewrite the Lagrange equations associated with \(\phi_1, \ldots, \phi_m\) as
\[
\phi_j = K_w \ast \left(\sum_{i=1}^{m} b_{ij} |\phi_i|^p |\phi_j|^{p-2} \phi_j\right),
\]
where
\[
K_w(x) = \frac{1}{2\sqrt{w}} e^{-\sqrt{w}|x|}.
\]
Since the convolution of \(K_w\) with a function that is everywhere non-negative and not identically zero gives an everywhere positive function, it follows that \(\phi_j(x) > 0\) for all \(x \in \mathbb{R}\) and they are all infinitely differentiable. Moreover, it follows from the proof of Theorem 8.1.1 in [7] that the functions \(\phi_j(x)\) are all exponentially decreasing and the fact that \((\phi_1, \ldots, \phi_m)\) are radially symmetric with respect to the same point \(x_0 \in \mathbb{R}\), (that is \(\phi_j(x) = \psi_j(|x - x_0|)\) for all \(j = 1, 2, \ldots, m\), where \(\psi_j\) are all radially symmetric) follows from [5].

Let \(\alpha_j(x) = \frac{u_j(x)}{\alpha_j(x)}\), then \(|\alpha_j(x)| = 1\) for all \(x \in \mathbb{R}\). It is easy to see that the real part of \(\pi_j(x)\alpha_j'(x)\) is vanishing. We therefore obtain that \(\alpha_j(x)\) is a constant due to the equality \(\|(u_j)_x\|^2 = \|\phi_j\|^2\). The statement (3) is then proved. Thus, Theorem 2.1 is true for \(m = k + 1\).

Since the functionals \(E\) and \(Q\) are all invariant under translations, an immediate consequence of the above result is that the set of minimizers \(G(\Lambda)\) is stable. (See, for example, [17]). Thus, the statement (4) is also clear.

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