1. What is a continuous random variable? What is the probability density function of a continuous random variable? What is the cumulative distribution function for a continuous random variable? How is it related to the density function?

\[ X \text{ is continuous if there exists a function } F : \mathbb{R} \to [0,1] \text{ such that for each } a,b \quad P(a \leq X \leq b) = \int_a^b f(x) \, dx \]

\[ F(x) = P(X \leq x) \text{ and } F'(x) = f(x) \]

2. What is the expected value and variance of a random variable?

\[ E(X) = \sum_x x P(x) \text{ if } X \text{ is discrete} \]

\[ \int_{-\infty}^{\infty} x f(x) \, dx \text{ if } X \text{ is continuous} \]

\[ \text{Var } X = E(X - E(X))^2 \]

3. What is meant by the memoryless property for an exponential distribution?

\[ P(X > s + t \mid X > s) = P(X > t) \]

4. Suppose you know the distribution for a random variable \( X \). How do you find the distribution of the random variable \( Y = g(X) \)?

\text{2-step method}

5. If \( X \) is a normal distribution with parameters \( \mu_x \) and \( \sigma_x \), how do you compute probabilities involving \( X \)?

\[ P(a < X < b) = P \left( \frac{a - \mu}{\sigma} < \frac{X - \mu}{\sigma} < \frac{b - \mu}{\sigma} \right) = P \left( \frac{a - \mu}{\sigma} < Z < \frac{b - \mu}{\sigma} \right) \]

6. How are the Poisson, Exponential, and Gamma distributions related?

Time until 1st event in a Poisson process is an exponential r.v. Exponential is a gamma.

Time until \( n \)-th event in a Poisson process is a gamma distribution.
7. How is the Gamma function a generalization of the factorial function?
\[ \Gamma(n) = (n-1)! \]

8. Why do normal distributions arise so frequently in applications?
Many measurements are normally distributed or the central limit theorem applies.

9. How do you compute the probability of an event that is described in terms of two random variables X and Y? How do you generalize the theory of random variables to higher dimensions?
See textbook. For higher dimension, we need linear algebra.

10. What is the joint cumulative distribution function for two random variables X and Y? What is the joint probability frequency function for two discrete random variables X and Y? What is the joint probability density function for two continuous random variables X and Y? How is it related to the joint cdf? Given the joint density function for X and Y, how do you obtain the marginal densities? Given random variables X and Y, how do you find the distribution of Y given X?
\[ F_{X,Y}(a,b) = P(X \leq a, Y \leq b) \]  
\[ f_{X,Y}(a,b) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y} \]
\[ P(\{X, Y\} \in D) = \iint_D f(x,y) \, dx \, dy \]  
\[ f_X(x) = \int_{-\infty}^{\infty} f(x,y) \, dy \]  
\[ f_Y(y) = \int_{-\infty}^{\infty} f(x,y) \, dx \]  
\[ f_Y(y \mid x) = \frac{f(x,y)}{f_X(x)} \]

11. How do you determine if two random variables are independent?
\[ f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) \]
\[ f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) \]

12. Suppose you know the joint density function for the random variables X and Y. How do you find the distribution for the random variable g(X, Y)?

2-step method
Review Problems: Test 2

1. Let \( f(x) = cx^2 \) for \( 0 < x < 2 \).

   a) Find the value of \( c \) that makes \( f(x) \) a density function for a random variable \( X \).

      \textit{covered for T1}

   b) Find the cumulative distribution function for \( X \).

      \textit{covered for T1}

   c) Find \( E(X) \) and \( \text{Var}(X) \).

      \textit{Later}

   d) Find \( P(X^2 < 2) \)

      \textit{covered for T1}

2. a) A point is chosen at random on a line segment of length 5. The line segment is then cut at this point and divided into two segments. Find the probability that the ratio of the shorter segment to the longer segment is less than \( \frac{3}{4} \).

   \textit{T1}

   b) Buses arrive at a specified stop at 15-minute intervals starting at 7 A.M. That is, they arrive at 7, 7:15, 7:30, 7:45, and so on. If a passenger arrives at the stop at a time that is uniformly distributed between 7 and 7:30, find the probability that he waits less than 5 minutes for a bus. Find the probability that he waits more than 10 minutes.

   \textit{T1}
3. Suppose $X$ has a uniform distribution on $[-2, 2]$ and $Y = X^2$. Find the density function for $Y$.

$$f_Y(x) = \frac{1}{4}, \quad -2 \leq x \leq 2.$$ 

$$F_Y(t) = P(Y \leq t) = P(X^2 \leq t) = P(-\sqrt{t} \leq X \leq \sqrt{t})$$

$$= \frac{\sqrt{t}}{4} = \frac{\sqrt{t}}{2}$$

$$F_Y(t) = f_Y(t) = \frac{1}{4} \cdot \frac{1}{\sqrt{t}} = \frac{1}{4\sqrt{t}}, \quad 0 \leq t \leq 4$$

4. Suppose the random variable $X$ is the length of life (in hours) of an electronic component and that $X$ has an exponential probability density function with mean 500 hours. Find the probability that the component lasts at least 800 hours. Suppose the component has been in operation for 300 hours. What is the probability that it will last another 800 hours?

$$T1$$

5. If the radius of a circle is an exponential random variable, find the density function for the area of the circle.

Let $X =$ radius of circle. $f_X(x) = \lambda e^{-\lambda x}, \quad t \geq 0$

Let $Y =$ area of circle; $Y = \pi x^2$

$@ F_Y(y) = P(Y \leq y)$

$@ f_Y(t) = F_Y'(t)$

6. Mathematics scores for USU students on the ACT test are normally distributed with mean 20 and standard deviation 4.

a) Find the probability that a randomly selected student has an ACT math score < 30.

$$P(X < 30) = P\left(\frac{X - 20}{4} < \frac{30 - 20}{4}\right) = P\left(Z < 0.15\right) = 0.062$$

b) What ACT score represents the 75-th percentile?

$$P\left(Z \leq .68\right) \approx .75$$

$$P\left(\frac{X - 20}{4} \leq .68\right) \approx .75$$

Solve $\frac{X - 20}{4} = .68$

$$X = 22.72$$

$X \approx 23$
7. Suppose $X$ is a standard normal random variable. Let $Y = X^2$. Find the density function for $Y$ and show that it is a Gamma density.

8. Let $X$ be the minimum of the two values when a pair of dice is rolled.

   a) Find the probability frequency function for $X$.

   $\begin{array}{c|cccccc}
   X & 1 & 2 & 3 & 4 & 5 & 6 \\
   P(X) & \frac{1}{36} & \frac{2}{36} & \frac{3}{36} & \frac{4}{36} & \frac{5}{36} & \frac{6}{36} \\
   \end{array}$

   b) Find the mean, variance, and standard deviation of $X$.

   later

9. Three players play ten rounds of a game, and each player has probability $1/3$ of winning each round. Find the joint distribution of the number of games won by each of the three players.

   $X =$ number won by player 1
   $Y =$
   $Z =$

   $P(L, J, K) = P(X=L, Y=j, Z=k) = \binom{n}{l, j, k} \left(\frac{1}{3}\right)^n \quad l + j + k = 10$

10. In the gambling game Chuck-a-Luck, for a $1 bet it is possible to win $1, $2, or $3 with respective probabilities 75/216, 15/216, 1/216. One dollar is lost with probability 125/216. If $X$ equals the payoff for this game, find $E(X)$ and $\text{Var}(X)$. Note that when a bet is won, the $1 that was bet is returned to the better in addition to the money won.

   later

11. A biased coin with probability of heads equal to $2/3$ is tossed until a head appears or 3 tosses have been made. Let $X$ denote the number of tosses. Find the mean and variance of $X$.

   later
17. Suppose $X, Y$ are continuous random variables with joint density

$$f(x, y) = \begin{cases} \frac{1}{2} x e^{-y} & \text{for } 0 < x < 2, \ y > 0 \\ 0 & \text{otherwise} \end{cases}$$

a) Find $P(X \leq 1, Y \leq 2)$.

$$ = \int_0^2 \int_0^{\frac{x}{2}} xe^{-y} \, dy \, dx$$

b) Find $f_x(x)$ and $f_y(y)$. Are $X, Y$ independent?

$$f_x(x) = \int_{-\infty}^{\infty} f(x,y) \, dy = \int_0^{\infty} \frac{1}{2} x e^{-y} \, dy = \frac{x}{2}$$

$$f_y(y) = \int_{-\infty}^{\infty} f(x,y) \, dx = \int_0^{2} \frac{1}{2} e^{-y} \, dx = e^{-y}$$

Are $X, Y$ independent? Yes, $f_x(x)$ and $f_y(y)$ are independent functions.

c) Find $P(X^2 + Y^2 \leq 4)$. Write your answer as an integral; do not evaluate.

$$ = \int_0^2 \int_0^{\sqrt{4-x^2}} \frac{1}{2} xe^{-y} \, dy \, dx$$

18. The joint probability density function of $X$ and $Y$ is given by

$$f(x, y) = e^{-(x+y)} \text{ for } 0 \leq x < \infty, \ 0 \leq y < \infty$$

Find $P(X < Y)$, $f_x(x)$, and $f_y(y)$.

$$f_x(x) = \int_{-\infty}^{\infty} e^{-x} e^{-y} \, dy = e^{-x}$$

$$f_y(y) = \int_{-\infty}^{\infty} e^{-x} e^{-y} \, dx = e^{-y}$$

$$P(X < Y) = \int_0^\infty \int_0^x e^{-x} e^{-y} \, dy \, dx$$
12. The joint density of $X$ and $Y$ is $f(x,y) = x + y$ for $0 < x < 1, 0 < y < 1$.

Find $f_x(x)$ and $f_y(y)$. Are $X$ and $Y$ independent? Find $P(X + Y < 1)$. Find $P(X > Y)$.

Find $P(Y > X^2)$.

\[
\begin{align*}
F_X(x) &= \int_0^x f(x,y) \, dy = x + \frac{1}{2} \\
F_Y(y) &= \int_0^y f(x,y) \, dx = y + \frac{1}{2} \\
\end{align*}
\]

$\not$ independent

\[
P(X + Y < 1) = \int_0^1 \int_0^{1-x} (x+y) \, dy \, dx
\]

\[
P(X > Y) = \int_0^1 \int_x^1 (x+y) \, dy \, dx
\]

13. The joint probability density function of $X$ and $Y$ is given by

\[
f(x,y) = \frac{1}{8} (y^2 - x^2) e^{-y} \text{ for } -y \leq x \leq y, \ 0 < y < \infty
\]

Write the density of $X$ and the density of $Y$ as integrals with the appropriate limits of integration. Do not evaluate the integrals. Find

\[
\begin{align*}
F_X(x) &= \int_{-\infty}^x f(x,y) \, dy = \int_{1-x}^0 f(x,y) \, dy \\
F_Y(y) &= \int_{-\infty}^y f(x,y) \, dx = \int_y^\infty f(x,y) \, dx \\
\end{align*}
\]

14. Suppose that the joint density of $X$ and $Y$ is given by

\[
f(x,y) = \frac{e^{-x/y} e^{-y}}{y} \text{ when } 0 < x < \infty, \ 0 < y < \infty.
\]

a) Find the conditional density of $X$ given that $Y = y$.

\[
f_X(x | Y = y) = \frac{\int_{-\infty}^x f(x,y) \, dx}{\int_{-\infty}^\infty f(x,y) \, dx} = \frac{e^{-\frac{x}{y}} e^{-y}}{y} = \frac{e^{-\frac{x}{y}}}{y}, \ x > 0
\]
b) Find $P(X > 1 \text{ given } Y = y) = \int_1^\infty \frac{e^{-\frac{x}{y}}}{y} \, dx$

15. A random variable $X$ has the density function

$$f(x) = \begin{cases} 3e^{-3x} & \text{if } 0 \leq x < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Find $E(e^X)$. Describe how you would compute the var $X$.

\textit{later}\textit{ later}

16. Let $X$ be a continuous random variable with probability density function

$$f(x) = \begin{cases} \frac{2}{x^2} & \text{if } 1 < x < 2 \\ 0 & \text{otherwise.} \end{cases}$$

Find $E(X)$ and var($X$).

\textit{later}