### Series Tests for Convergence/Divergence

<table>
<thead>
<tr>
<th>Test</th>
<th>Suppositions</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Test for Convergence</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>If you can see that ( \lim_{n \to \infty} a_n \neq 0 ), use Test</td>
<td>( \sum_{n=1}^{\infty} a_n ) converges</td>
<td></td>
</tr>
<tr>
<td>For Convergence</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Integral Test</strong></td>
<td>Suppose function ( f(x) ) is</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• continuous on ([k, \infty))</td>
<td>( \int_{k}^{\infty} f(x) , dx ) converges ( \Rightarrow ) ( \sum_{n=k}^{\infty} a_n ) converges</td>
</tr>
<tr>
<td></td>
<td>• positive on ([k, \infty))</td>
<td>( \int_{k}^{\infty} f(x) , dx ) diverges ( \Rightarrow ) ( \sum_{n=k}^{\infty} a_n ) diverges</td>
</tr>
<tr>
<td></td>
<td>• ultimately decreasing ( f'(x) &lt; 0 ) for ( x &gt; N )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• ( a_n = f(n) )</td>
<td></td>
</tr>
<tr>
<td><strong>Comparison Test</strong></td>
<td>Suppose ( \sum a_n ) and ( \sum b_n ) have positive terms for all ( n \geq N ). ( \sum b_n ) is a known series, usually the p-series or geometric series.</td>
<td>( \sum b_n ) converges and ( a_n \leq b_n ) ( \forall \ n \geq N ) ( \Rightarrow ) ( \sum a_n ) converges</td>
</tr>
<tr>
<td></td>
<td>Note: ( \sum b_n ) is a known series, usually the p-series or geometric series.</td>
<td></td>
</tr>
<tr>
<td><strong>Limit Comparison Test</strong></td>
<td>Suppose ( \sum a_n ) and ( \sum b_n ) have positive terms for all ( n \geq N ). ( \sum b_n ) is a known series, usually the p-series or geometric series.</td>
<td>( \lim_{n \to \infty} \frac{a_n}{b_n} = c ) (where ( c ) is finite and ( c &gt; 0 )) ( \Rightarrow ) both series converge or both series diverge</td>
</tr>
<tr>
<td></td>
<td>Note: ( \sum b_n ) is a known series, usually the p-series or geometric series.</td>
<td></td>
</tr>
<tr>
<td><strong>Ratio Test</strong></td>
<td>Suppose ( \sum a_n ) has positive terms</td>
<td></td>
</tr>
<tr>
<td></td>
<td>The Ratio Test is a convenient test for series that involve factorials or other products (including a constant raised to the ( n )th power).</td>
<td>( (i) ) if ( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = L &lt; 1 ) ( \Rightarrow ) ( \sum a_n ) is absolutely convergent</td>
</tr>
<tr>
<td></td>
<td>( (ii) ) if ( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = L &gt; 1 ) ( \Rightarrow ) ( \sum a_n ) diverges</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( (iii) ) if ( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1 ) ( \Rightarrow ) test is inconclusive</td>
<td></td>
</tr>
<tr>
<td><strong>Root Test</strong></td>
<td>Suppose ( \sum a_n ) has positive terms for ( n \geq N )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( (i) ) if ( \lim_{n \to \infty} \sqrt[n]{a_n} = \rho &lt; 1 ) ( \Rightarrow ) ( \sum a_n ) converges</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( (ii) ) if ( \lim_{n \to \infty} \sqrt[n]{a_n} = \rho &gt; 1 ) ( \Rightarrow ) ( \sum a_n ) diverges</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( (iii) ) if ( \lim_{n \to \infty} \sqrt[n]{a_n} = 1 ) ( \Rightarrow ) test is inconclusive</td>
<td></td>
</tr>
<tr>
<td><strong>Alternating Series Test</strong></td>
<td>Given an alternating series ( \sum_{n=1}^{\infty} (-1)^{n-1} b_n ) ( (b_n &gt; 0) ) ( \Rightarrow ) ( \sum_{n=1}^{\infty} (-1)^{n-1} b_n ) converges</td>
<td></td>
</tr>
<tr>
<td>Use for series of the form ( \sum_{n=1}^{\infty} (-1)^{n-1} b_n ).</td>
<td>( (i) ) if ( b_{n+1} \leq b_n ) ( \Rightarrow ) ( \sum_{n=1}^{\infty} (-1)^{n-1} b_n ) converges</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( (ii) ) if ( \lim_{n \to \infty} b_n = 0 ) ( \Rightarrow ) ( \sum_{n=1}^{\infty} (-1)^{n-1} b_n ) converges</td>
<td></td>
</tr>
</tbody>
</table>

**Definitions**

- A series \( \sum a_n \) converges absolutely (is absolutely convergent) if the corresponding series of absolute values, \( \sum |a_n| \), converges.
- A series that converges but does not converge absolutely converges conditionally.

**Absolute Convergence Theorem**

If \( \sum |a_n| \) converges, then \( \sum a_n \) converges.
Theorem 1 — Limit Laws
Let \( \{a_n\} \) and \( \{b_n\} \) be sequences of real numbers and let \( A \) and \( B \) be real numbers. The following rules hold if \( \lim_{n \to \infty} a_n = A \) and \( \lim_{n \to \infty} b_n = B \).

1. Sum Rule: \( \lim_{n \to \infty} (a_n + b_n) = A + B \).
2. Difference Rule: \( \lim_{n \to \infty} (a_n - b_n) = A - B \).
3. Product Rule: \( \lim_{n \to \infty} (a_n \cdot b_n) = A \cdot B \).
4. Constant Multiple Rule: \( \lim_{n \to \infty} (k \cdot b_n) = k \cdot B \) (any number \( k \)).
5. Quotient Rule: \( \lim_{n \to \infty} \left( \frac{a_n}{b_n} \right) = \frac{A}{B} \) if \( B \neq 0 \).

Theorem 2 — Sandwich Theorem for Sequences
Let \( \{a_n\} \), \( \{b_n\} \), and \( \{c_n\} \) be sequences of real numbers. If \( a_n \leq b_n \leq c_n \) holds for all \( n \geq N \), and if \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = L \), then \( \lim_{n \to \infty} b_n = L \).

Theorem 3 — Continuous Function Theorem for Sequences
Let \( \{a_n\} \) be a sequence of real numbers. If \( a_n \to L \) and if \( f \) is a function that is continuous at \( L \) and defined at all \( a_n \), then \( f(a_n) \to f(L) \).

Theorem 4
Suppose that \( f(x) \) is a function defined for all \( x \geq n_0 \) and that \( \{a_n\} \) is a sequence of real numbers such that \( a_n = f(n) \) for \( n \geq n_0 \). Then \( \lim_{x \to \infty} f(x) = L \implies \lim_{n \to \infty} a_n = L \).

Theorem 5 — Commonly Occurring Limits
The following six sequences converge to the limits listed below:

1. \( \lim_{n \to \infty} \frac{\ln n}{n} = 0 \)
2. \( \lim_{n \to \infty} \sqrt[n]{n} = 1 \)
3. \( \lim_{n \to \infty} x^{1/n} = 1 \) \((x > 0)\)
4. \( \lim_{n \to \infty} x^n = 0 \) \((|x| < 1)\)
5. \( \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x \) \((\text{any } x)\)
6. \( \lim_{n \to \infty} \frac{x^n}{n!} = 0 \) \((\text{any } x)\)

In Formulas (3) through (6), \( x \) remains fixed as \( n \to \infty \).

Known Series

The geometric series
\[
\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \ldots
\]
is convergent if \(|r| < 1\) and its sum is
\[
\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r} \quad |r| < 1
\]
If \(|r| \geq 1\), the geometric series is divergent.

The p-series
\[
\sum_{n=1}^{\infty} \frac{1}{n^p}
\]
is convergent if \( p > 1 \) and divergent if \( p \leq 1 \).

The harmonic series
\[
\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots
\]
is divergent.

Telescoping series

Series Definitions / Theorem

Definitions
Given a sequence of numbers \( \{a_n\} \), an expression of the form
\[
a_1 + a_2 + a_3 + \ldots + a_n + \ldots
\]
is an infinite series. The number \( a_n \) is the \( n \)th term of the series. The sequence \( \{s_n\} \) defined by
\[
s_1 = a_1
s_2 = a_1 + a_2
\vdots
s_n = a_1 + a_2 + \ldots + a_n = \sum_{k=1}^{n} a_k
\]
is the sequence of partial sums of the series, the number \( s_n \) being the \( n \)th partial sum. If the sequence of partial sums converges to a limit \( L \), we say that the series converges and the sum is \( L \). In this case, we also write
\[
a_1 + a_2 + \ldots + a_n + \ldots = \sum_{n=1}^{\infty} a_n = L
\]
If the sequence of partial sums of the series does not converge, we say that the series diverges.

Theorem 7: If the series \( \sum_{n=1}^{\infty} a_n \) is convergent, then \( \lim_{n \to \infty} a_n = 0 \).
(Note: The converse of this theorem is not true in general. If \( \lim_{n \to \infty} a_n = 0 \), we cannot conclude that \( \sum_{n=1}^{\infty} a_n \) is convergent. The harmonic series is an example.)