Monotonic Functions and the 1st Derivative Test

Four important consequences of the Mean Value Theorem:

1. If $f'(x)>0$ for each x in (a,b) then f is (strictly) increasing on (a,b).

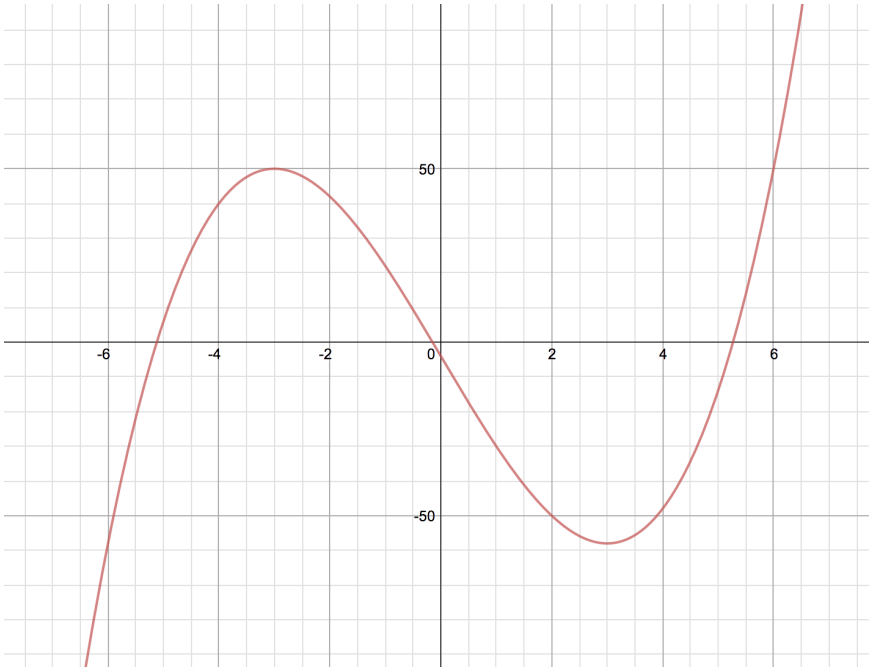
2. If $f'(x)<0$ for each x in (a,b) then f is (strictly) decreasing on (a,b).

3. If $f'(x)=0$ for each x in (a,b) then f is a constant function on (a,b).

4. If $f'(x) = g'(x)$ for each x in (a,b) then there exists a constant c such that $f(x) = g(x) + c$ for each x in (a,b) .

<u>Example</u>

Find the critical points of the function f defined by $f(x)=x^3-27x-4$ and determine the intervals on which f is increasing and the the intervals on which f is decreasing. Can you determine the local extreme values from this analysis?



First Derivative Test:

Suppose $x = c$ is a critical point for f .

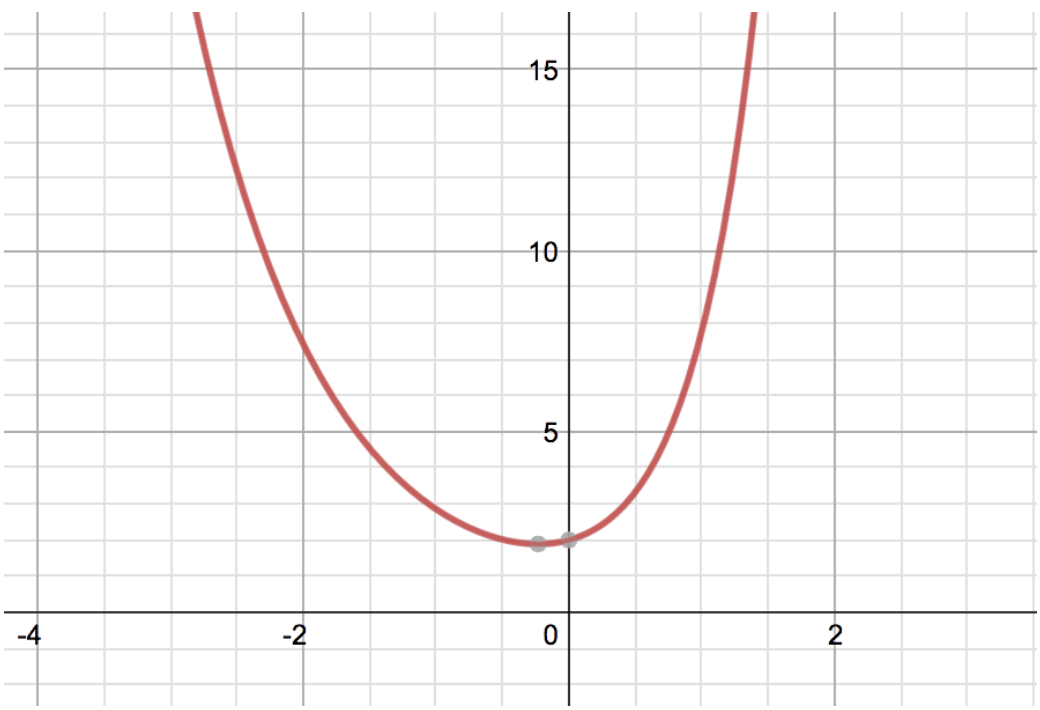
If f' changes from negative to positive at c then f has a local minimum at c .

If f' changes from negative to positive at c then f has a local maximum at c .

If f' does not change sign at c then f has no local extremum at c .

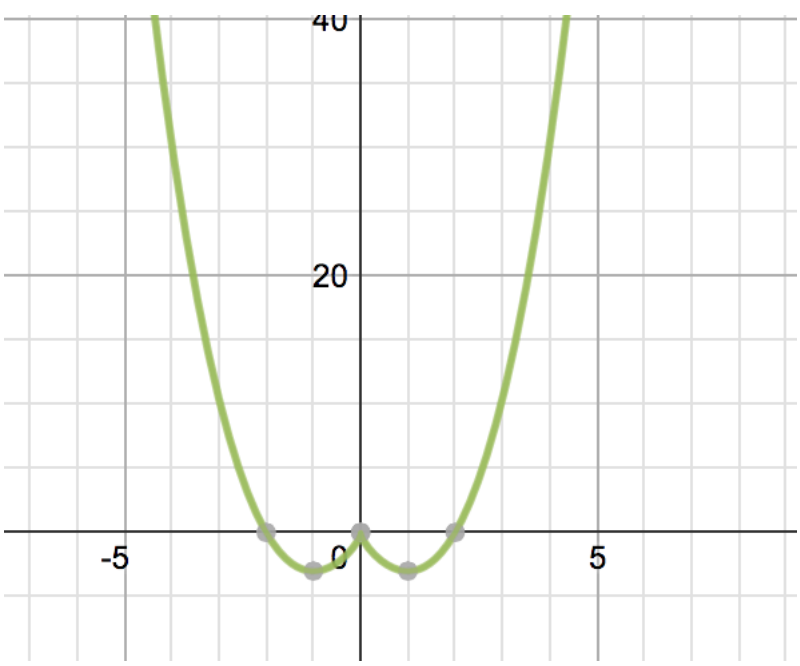
Example

Let f be defined by $f(x) = e^{2x} + e^{-x}$. Find the local extreme values and sketch the graph.



Example

Let $w(x)=x^{\frac{2}{3}}(x^2-4)$. Find the intervals on which w is increasing and the intervals on which w is decreasing. Find the local extreme values and sketch the graph.



Concavity and Curve Sketching

<u>Example</u>

Consider the function f defined by $f(x) = x^3$. Note that

$f'(x) = 3x^2$, $f''(x) = 6x$

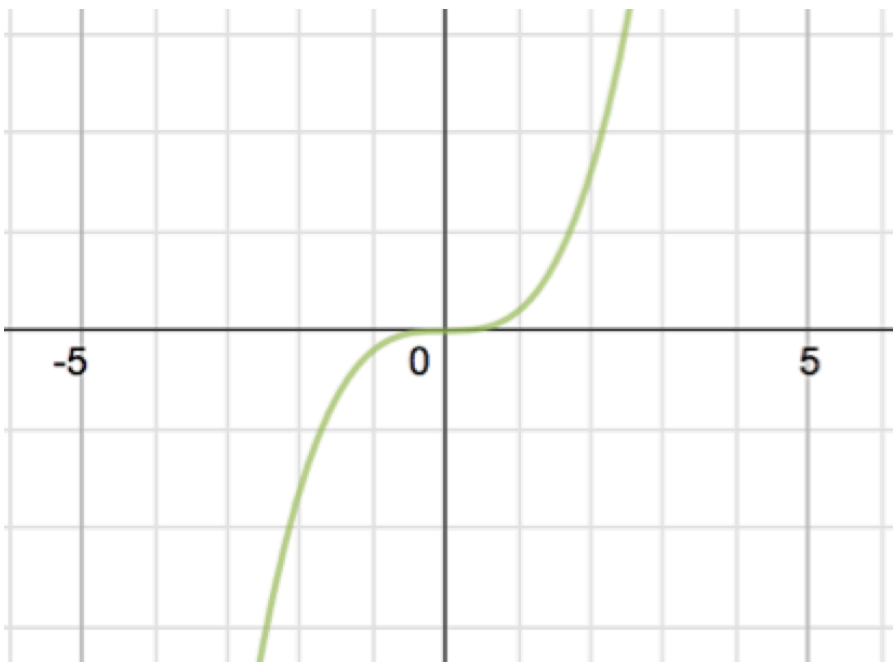
$f''(x) > 0 \quad for \quad x > 0$

$f''(x) < 0 \quad for \quad x < 0$

f' is increasing on $(0, \infty)$

f' is decreasing on $(-\infty, 0)$

f'' changes sign at $x = 0$



<u>Definition</u>

The function f is concave up on the interval (a, b) if f' is increasing on (a, b).

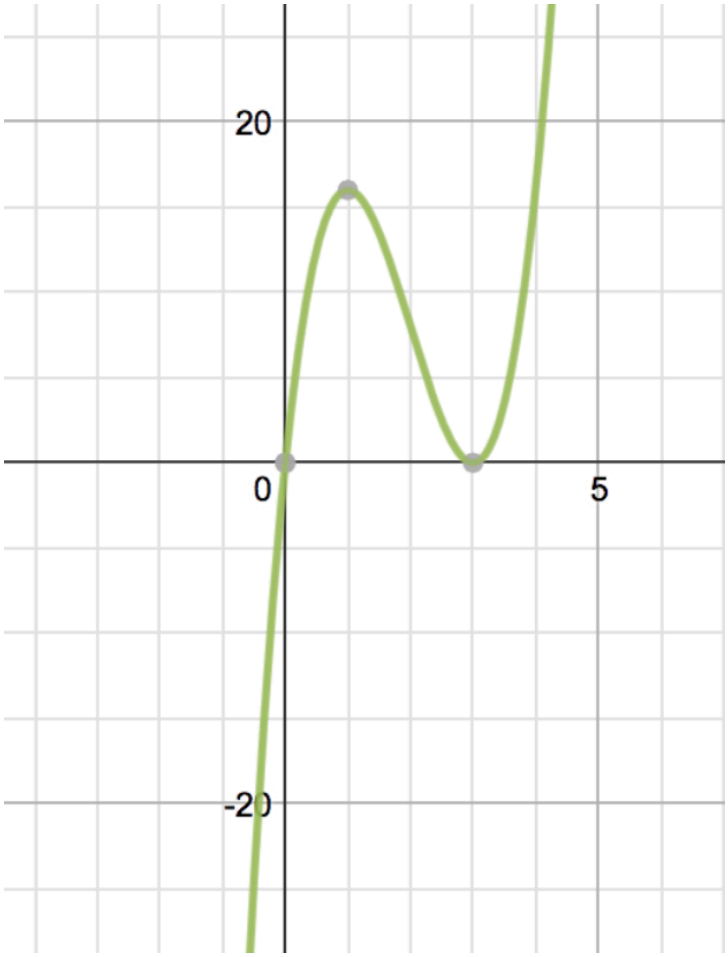
The function f is concave down on the interval (a, b) if f' is decreasing on (a, b).

Definition

$(c, f(c))$ is a point of inflection if the graph of f has a tangent line at $(c, f(c))$ and the concavity of f changes at $(c, f(c))$.

Example

Let $f(x) = x(6-2x)^2$. Identify all extreme values, the intervals where f is increasing or decreasing, where it is concave up, where it is concave down, its inflection points, and sketch the graph.



<u>Second Derivative Test</u>

Suppose f'' is continuous near $x=c$.

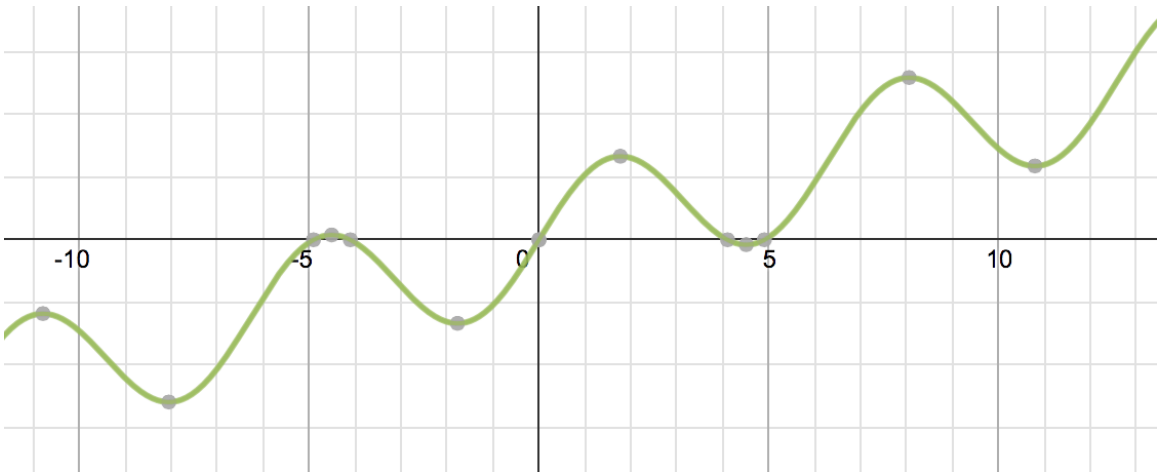
1. If $f'(c)=0$ *and* $f''(c)<0$ then f has a local maximum at $x=c$.

2. If $f'(c)=0$ *and* $f''(c)>0$ then f has a local minimum at $x=c$.

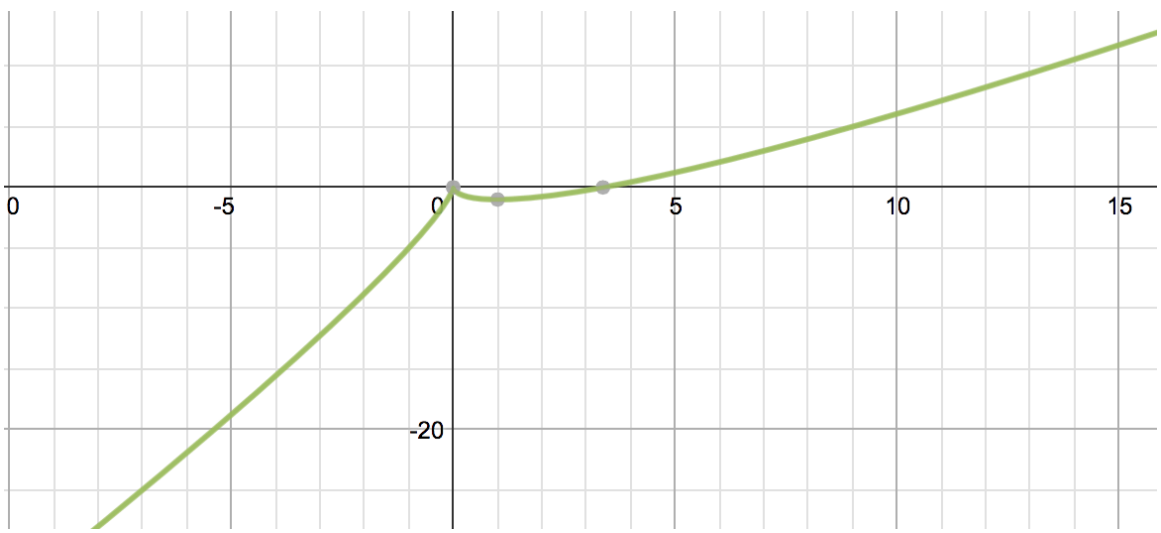
3. If $f'(c)=0$ *and* $f''(c)=0$ then the test fails.

Sketch the graphs of the following:

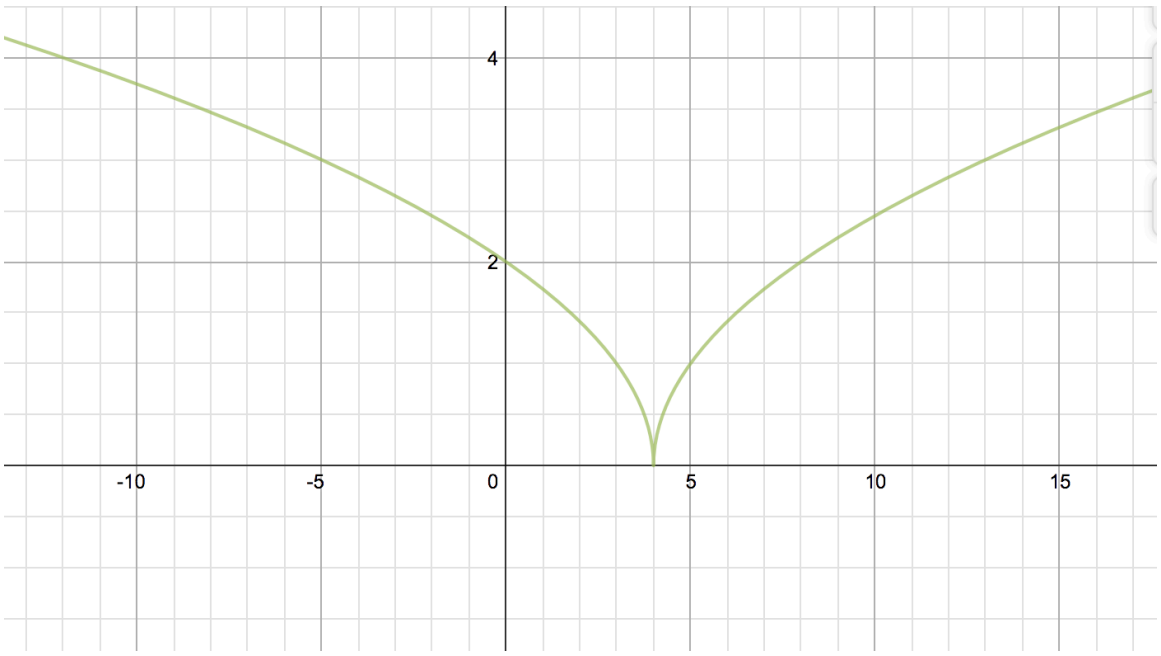
a) $f(x) = x + 5\sin x$



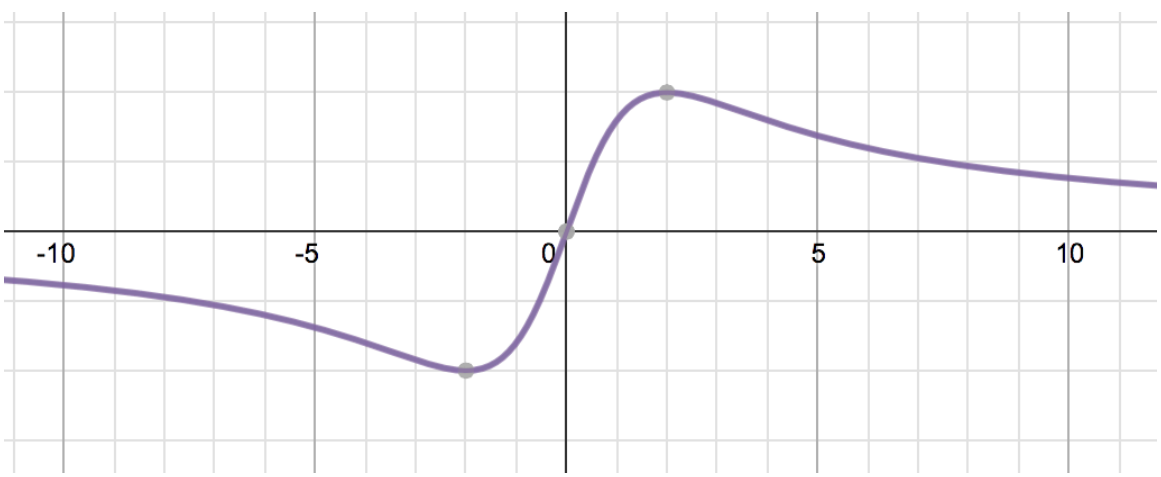
b) $g(x) = 2x - 3x^{2/3}$



c) $h(x)=\sqrt{|x-4|}$



d) $k(x) = \dfrac{4x}{x^2+4}$



e) $s(x) = \dfrac{x-1}{x^2(x-2)}$