4.5 MODELS FOR GROWTH, DECAY, AND CHANGE

I find that I may have emphasized the need to escape from the devils of mathematics to embark on the pleasures of the real world. But it works both ways, and sometimes the devils of the real world drive one into the pleasures of studying mathematics.

Cathleen S. Morawetz

Exponential and logarithmic functions are used to model many real-world processes, some of which we mentioned in earlier sections. In this section we look at additional applications.

I developed a proficiency [in junior high school] in simple algebra that lasted for a long time and has been very useful. My mother gave me her college algebra book. I learned from it how to solve word problems, although I remember distinctly that I never really understood them. I could do only the problems that followed the pattern of the examples in the book. My view is that if you have a firm grasp of technique, you can then concentrate on theory without having to think about the technical details.
Ralph P. Boas, Jr.

Exponential Growth

When scientists measure population size, they see regular changes. Whether they study fish, bacteria, or mammals, they observe that the rate of change is proportional to the number of organisms present; with more bacteria in a culture, colonies grow faster (as long as there is adequate food). A similar kind of growth occurs in a financial setting with compound interest. The amount of interest depends on the amount of money invested, and a larger investment grows faster.

We learn in calculus that exponential functions can model any kind of growth for which the rate of change is proportional to the amount present. Hence this kind of growth is called **exponential growth.**

To express exponential growth mathematically, suppose $A(t)$ denotes the amount of substance or the number of organisms present at time t. Then $A(t)$ is given by

$$A(t) = Ce^{kt},$$

where C and k are constants. When t is 0, the formula gives $A(0) = Ce^{k \cdot 0}$, or $A(0) = C$. Hence, for any exponential growth, C is the amount present at the time measurement begins, when t is 0; we replace C by A_0.

Exponential growth formula

Suppose the rate of change of some substance or quantity is proportional to the amount present, then the amount or number $A(t)$ at time t is given by

$$A(t) = A_0 e^{kt} \qquad \textbf{(1)}$$

where A_0 is the initial amount (the amount present when t is 0), and k is a positive constant determined by the particular substance.

In many problems the constant k is determined experimentally. For instance, a scientist may find that the number of bacteria in a culture doubles every 72 minutes. This information is enough to determine the value of k, as shown in the following example.

Strategy: Use Equation (1) with $A_0 = 500$ and $A(1.2) = 1000$, since 72 minutes is 1.2 hours. Find k.

▶EXAMPLE 1 ***Exponential growth*** A sample culture medium contains approximately 500 bacteria when first measured, and 72 minutes later the number has doubled to 1000.

(a) Determine a formula for the number $A(t)$ at any time t hours after the initial measurement.
(b) What is the number of bacteria at the end of 3 hours?
(c) How long does it take for the number to increase tenfold to 5000?

Solution

(a) Follow the strategy. When t is 1.2, Equation (1) becomes

$$1000 = 500e^{k(1.2)}.$$

Divide by 500, take the natural logarithm of both sides, and solve for k.

$$2 = e^{1.2k}, \qquad \ln 2 = \ln e^{1.2k} = 1.2k, \qquad k = \frac{\ln 2}{1.2} \approx 0.578.$$

Replacing k by 0.578 and A_0 by 500 gives the desired equation.

$$A(t) = 500e^{0.578t} \tag{2}$$

(b) When t is 3, $A(3) = 500e^{0.578(3)} = 500e^{1.734} \approx 2832$, so at the end of 3 hours there are approximately 2800 bacteria in the culture.

(c) To find t when $A(t)$ is 5000, substitute 5000 for $A(t)$ in Equation (2) and solve for t.

$$5000 = 500e^{0.578t} \qquad 10 = e^{0.578t}$$

$$\ln 10 = \ln e^{0.578t} = 0.578t$$

$$t = \frac{\ln 10}{0.578} \approx 3.98$$

It takes about 4 hours for the number of bacteria to increase tenfold.

Graphical Draw a graph of $A(x) = 500e^{0.578x}$ in $[0, 5] \times [500, 5100]$. Then trace and zoom as needed to see that when $x = 3$, $A = 2832$, and when $A = 5000$, $x = 3.98$. ◀

Compound and Continuous Interest

If money is invested in an account that pays interest at a rate r compounded n times a year, the growth is not described by Equation (1). We need another formula. When the annual interest rate is given as a percentage, we express r as a decimal; for a rate of 6 percent we write $r = 0.06$.

Compound interest formula

Suppose A_0 dollars are invested in an account that pays interest at rate r compounded n times a year. The number of dollars $A(t)$ in the account t years later is given by

$$A(t) = A_0\left(1 + \frac{r}{n}\right)^{nt}. \tag{3}$$

Compound interest is paid only at the end of each compounding period. If interest is compounded quarterly, then the interest is credited at the end of each three-month period. To apply Equation (3) for other values of t, we should replace the exponent nt by the greatest integer $[nt]$.

As the number of times a year that interest is compounded increases, we approach what is called **continuous compounding.** To see what happens to $A(t)$ as n becomes large ($n \to \infty$), replace $\frac{r}{n}$ by x and rewrite the exponent nt as $(\frac{n}{r})(rt)$, or $(\frac{1}{x})(rt)$, so Equation (3) becomes

$$A(t) = A_0[(1 + x)^{1/x}]^{rt}. \tag{4}$$

Now, as $n \to \infty$, $\frac{r}{n} \to 0$, so $x \to 0$. We are interested in what happens to the expression $(1 + x)^{1/x}$ as $x \to 0$. This is equivalent to the problem we considered in Section 4.1. See Exercise 38, where you are asked to show that $(1 + x)^{1/x} \to e$ as $x \to 0$. Thus, when interest is compounded continuously at rate r for t years, *compound interest becomes exponential growth.* Equation (4) becomes $A(t) = A_0 e^{rt}$.

Continuous interest formula

Suppose A_0 dollars are invested in an account that pays interest at rate r compounded continuously. Then the number of dollars $A(t)$ in the account t years later is given by

$$A(t) = A_0 e^{rt}. \quad \textbf{(5)}$$

▶EXAMPLE 2 Compound interest Suppose \$2400 is invested in an account in which interest is compounded twice a year at the rate of 8 percent.

(a) How much is in the account at the end of ten years?
(b) How long does it take to double the initial investment?
(c) Answer the same questions if the money is compounded continuously.
(d) Draw graphs of Y1 = 2400 (1.04) ^ (2X) (interest compounded twice a year) and Y2 = 2400 e ^ (.08X) (continuous compounding) on the same screen $[0, 12] \times [2400, 5500]$. How soon does the continuous interest curve become visibly higher? Trace and zoom as needed to answer questions **(a)** and **(b).**

Strategy: For **(a)** and **(b)**, replace A_0 by 2400, r by 0.08, and n by 2 in Equation (3), then use the resulting equation. For **(c)**, replace A_0 by 2400 and r by 0.08 in Equation (5), then use the resulting equation.

Solution

Follow the strategy.

$$A(t) = 2400(1 + 0.04)^{2t} = 2400(1.04)^{2t}, \text{ so } A(t) = 2400(1.04)^{2t}.$$

(a) In ten years, t is 10, so

$$A(10) = 2400(1.04)^{20} \approx 5258.70$$

At the end of ten years the account will be worth \$5258.70.

(b) Solve the following for t:

$$4800 = 2400(1.04)^{2t}, \qquad 2 = (1.04)^{2t},$$

$$\ln 2 = \ln(1.04)^{2t} = 2t \ln 1.04, \qquad \text{or} \qquad t = \frac{\ln 2}{2 \ln 1.04} \approx 8.8.$$

The \$2400 investment doubles in about 8 years and 10 months, but the account will not be credited with the last interest until the end of the year.

(c) Using Equation (5) instead of Equation (3),

$$A(t) = 2400e^{0.08t}.$$

In ten years, $A(10) = 2400e^{0.8} \approx 5341.30$, so continuous interest returns nearly \$83 more on a \$2400 investment than semiannual compounding over that time. To see how long it takes to double the investment, solve for t:

$$4800 = 2400e^{0.08t} \qquad t = \frac{\ln 2}{0.08} \approx 8.66.$$

The investment doubles in 8 years and 8 months.

(d) The graphs of the two functions are indistinguishable for the first half of the time interval (until x is about 6) even though when we trace, we can see that continuously compounded interest yields about \$36 more when $x = 6$. We can get the same answers from tracing along the graphs that we obtained above. It may help to adjust your window as suggested in the following Technology Tip, and in addition, to see how the interest is actually added to the account—at the end of each accounting period—graph Y1 = 2400 (1.04) ^ Int (2X) in dot mode rather than connected mode. ◀

TECHNOLOGY TIP ◆ ***"Nice-pixel" windows***

If you feel that nice pixel coordinates are helpful in reading information from a graph, use an x-range that is a multiple of your decimal window range. In the example above, we want something that includes [0, 12]. On the TI-82 and Casio fx–7700, the decimal window goes from −4.7 to 4.7, a total of 9.4 units or 94 tenths. If we multiply the number of pixel columns by 1.5, we have $9.4 \times 1.5 = 14.1$, so [−2, 12.1], works well on the TI-82. Similarly, [−2, 12.25] is good on the TI-81, and [0, 12.6] for the TI-85 or Casio *fx*-9700, [−1, 12] on the HP-38 and HP-48. The y-range must include [2400, 6200].

Exponential Decay

Certain materials, such as radioactive substances, decrease with time, rather than increase, with the rate of decrease proportional to the amount. Such negative growth is described by exponential functions, very much like exponential growth except for a negative sign in the exponent.

Exponential decay formula

Suppose the rate of decrease of some substance is proportional to the amount present. The amount $A(t)$ at time t is given by

$$A(t) = A_0 e^{-kt} \qquad \textbf{(6)}$$

where A_0 is the initial amount (the amount present when t is 0), and k is a positive constant determined by the particular substance.

►EXAMPLE 3 ***Radioactive decay*** Strontium-90 has a half-life of 29 years. Beginning with a 10 mg sample, **(a)** determine an equation for the amount $A(t)$ after t years and **(b)** find how long it takes for the sample to decay to 1 mg. **(c)** Check your result in part **(b)** by drawing a calculator graph of $A(t)$ in $[0, 100] \times [0, 8]$ and zooming in as needed.

Strategy: First, replace A_0 by 10 in Equation (6), then in the resulting equation use $A(29) = 5$ and solve for k. Using this value of k in Equation (6) gives the decay equation for strontium-90.

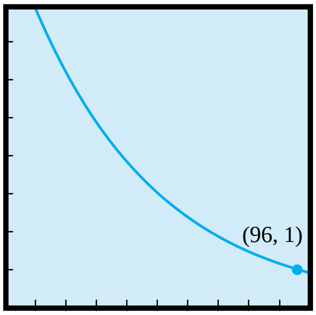
Solution

(a) Follow the strategy. $A(t) = 10e^{-kt}$, and in 29 years, half the sample will remain, so $A(29) = 5$. Substitute 29 for t and 5 for A, so

$$5 = 10e^{-29k} \quad \text{or} \quad e^{-29k} = \frac{1}{2}, \text{ so } -29k = \ln\left(\frac{1}{2}\right).$$

$$k = \frac{\ln\left(\frac{1}{2}\right)}{-29} = \frac{-\ln 2}{-29} \approx 0.0239.$$

Therefore, the decay equation for strontium-90 is $A(t) = 10e^{-0.0239t}$.



[0, 100] by [0, 8]
$A(t) = 10e^{-0.0239t}$

FIGURE 26

(b) To find when $A(t)$ is 1, replace $A(t)$ by 1 and solve the resulting equation for t.

$$1 = 10e^{-0.0239t}, \qquad \ln\left(\frac{1}{10}\right) = \ln e^{-0.0239t} = -0.0239t,$$

$$t = \frac{\ln\left(\frac{1}{10}\right)}{-0.0239} \approx 96.$$

It takes 96 years for 90 percent of the original amount of strontium-90 to decay.

(c) The graph of $A(t) = 10e^{-0.0239t}$ is shown in Figure 26. Observe that the function is decreasing. If we zoom in near the point where $A(t) = 1$, we find that $t \approx 96$. ◀

Hatchee Reservoir Revisited

The contamination of Hatchee Reservoir described in the chapter introduction is a classic case of exponential decay, where the water flowing through the reservoir flushes out half of the pollutants every fifteen days. According to Equation (6), the amount $A(t)$ of the toxic chemical left after t days is given by $A(t) = A_0e^{-kt}$, where we must determine the constant k.

▶EXAMPLE 4 ***Will Hatchee be clean by July 4?*** Find the constant k in the equation $A(t) = A_0e^{-kt}$ and determine how much of the toxic chemical will be left on July 4.

Solution

In fifteen days, $t = 15$, and so $A(15) = 0.5A_0$. Substituting these values into the equation for A, we have $0.5A_0 = A_0e^{-15k}$. Dividing by A_0 and taking the natural logarithm of both sides, we can solve for k:

$$\begin{aligned} 0.5 &= e^{-15k} \\ \ln 0.5 &= -15k \\ k &= (\ln 0.5)/(-15) \approx 0.0462. \end{aligned}$$

We store the entire display for computing, but we have $A(t) \approx A_0e^{-0.0462t}$.

As a check, when $t = 60$ (June 30), the formula gives $A(60) \approx 0.0625A_0$, confirming the simple analysis we gave in the introduction that 6.25% of the original contamination would remain on June 30. We can now determine the pollution level on July 4, when $t = 64$: $A(64) \approx 0.0520A_0$. Thus just *over* 5% will remain on July 4. City officials will have to decide whether the risk outweighs town tradition. Since the pollution level is predicted to be so near the declared safe level, it might pay the city to invest in more testing to see how accurately the mathematical model predicts the measured pollution level. ◀

Carbon Dating

Radioactive decay is used to date fossils. The method involves the element carbon. Carbon-12 is a stable isotope, while carbon-14 is a radioactive isotope with a half-life of approximately 5700 years. Fortunately for us, the concentration of C^{14} in the air we breathe and the food we eat is extremely small (about 10^{-6} percent).

Although C^{14} disintegrates as time passes, the amount of C^{14} in the atmosphere remains in equilibrium because it is constantly being formed by cosmic rays. All living things regularly take in carbon, and the proportion of C^{14} in living organisms reflects the proportion in the atmosphere. When an organism dies, however, the C^{14} is not replenished and the decay process decreases the ratio of C^{14} to C^{12}. By measuring this ratio in organic material, it is possible to determine the number of years since the time of death. The technique is known as *carbon dating* (see the Historical Note, "Exponential Functions, Dating, and Fraud Detection").

▶EXAMPLE 5 ***Dating Crater Lake*** A tree felled by the eruption that created Crater Lake in Oregon was found to contain 44 percent of its original amount of carbon-14. Use 5700 years as the half-life of carbon-14 and determine the age of Crater Lake.

Strategy: Crater Lake was formed when the tree died, so find how long the tree has been dead. Use Equation (6) and when t is 5700, $A(t)$ is $\frac{A_0}{2}$. Solve for k. Substitute this value in Equation (6) to get the decay equation for carbon-14. Now replace $A(t)$ by $0.44A_0$ and solve the resulting equation for t.

Solution

Follow the strategy.

$$A(t) = A_0 e^{-kt}$$

$$\frac{1}{2}A_0 = A_0 e^{-5700k} \qquad \text{or} \qquad \frac{1}{2} = e^{-5700k}$$

Take natural logarithms and use the fact that $\ln(\frac{1}{2}) = -\ln 2$:

$$-\ln 2 = \ln e^{-5700k} \qquad \text{or} \qquad -\ln 2 = -5700k$$

$$\text{or } k = \frac{\ln 2}{5700} \approx 0.0001216.$$

Therefore, the decay equation for carbon-14 is

$$A(t) = A_0 e^{-0.0001216t} \tag{7}$$

Since 44 percent of the original amount of carbon-14 still remained when the tree was discovered, find the value of t for which $A(t)$ is $(0.44)A_0$. Substitute $(0.44)A_0$ for $A(t)$ in Equation (7):

$$(0.44)A_0 = A_0 e^{-0.0001216t} \qquad \text{or} \qquad 0.44 = e^{-0.0001216t}$$

$$\ln 0.44 = \ln e^{-0.0001216t} \qquad \text{or} \qquad \ln 0.44 = -0.0001216t$$

$$t = -\frac{\ln 0.44}{0.0001216} \approx 6751.$$

Crater Lake was formed approximately 7000 years ago. ◀

TECHNOLOGY TIP ◆

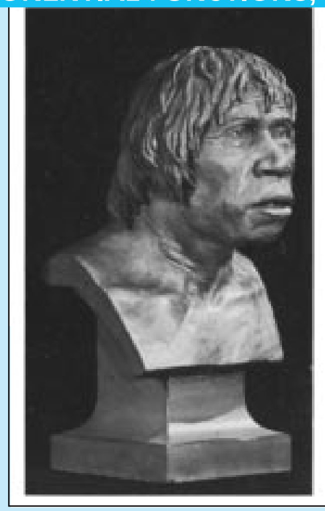
What about A_0?

The formula for $A(t)$ in Example 5 involves the initial amount, A_0. If we want to use graphical methods to solve the problem, once we have the decay equation, we can take any constant for the initial amount, say $A_0 = 1$. Then to find the value of t when 44% of the initial amount remains, trace along the curve $A(t) = e^{-0.0001216t}$ and find the t-value for which $A(t) \approx .44$.

HISTORICAL NOTE EXPONENTIAL FUNCTIONS, DATING, AND FRAUD DETECTION

The discovery of radiocarbon dating in 1949 by Willard F. Libby opened new ways to learn about the past. The half-life of carbon-14 allows dependable dating of organic material up to a range of 10,000 or 20,000 years.

Potassium allows dating on a much longer scale, albeit less precisely. Each of our bodies contains about a pound of potassium, including a miniscule fraction of radioactive potassium-40, which is changing (into argon gas) at a rate of about 500 atoms per second. Potassium-argon dating established the age of the fossil hominid Lucy at over 3 million years.



Bust of Piltdown Man

In 1908 bits of bone that comprised part of a human skull were found in a gravel pit in Piltdown, Sussex, England. Four years later part of an apelike jawbone showed up in the same location. Thus was born Piltdown Man, one of the strangest puzzles in human paleontology.

Joining a human cranium with an apelike jaw raised problems for students of human evolution and fueled a vigorous controversy that raged for years. Not until 1953 did fluorine dating (based on the fact that bones and teeth absorb fluorine from soil and groundwater at a constant rate) finally show that the cranium and jawbone did not belong together. The newly discovered radiocarbon dating showed that the skull dated from near Chaucer's time (about 600 years earlier—hardly prehistoric), and the jaw was even younger. It had belonged to an orangutan from the East Indies.

The whole Piltdown affair was perhaps the greatest hoax in the history of science. Professionals and amateurs alike (including Sir Arthur Conan Doyle, creator of Sherlock Holmes) became embroiled in the disputes. The identity of the perpetrators remains unresolved, but progress in the method of science, including mathematical dating analyses, helped to uncover the fraud.

The next example presents another illustration of exponential decay.

►EXAMPLE 6 ***Atmospheric pressure*** Standard atmospheric pressure at sea level is 1035 g/cm^2. Experimentation shows that up to about 80 km ($\approx$ 50 mi), the pressure decreases exponentially. The atmospheric pressure (in g/cm^2) at an altitude of h kilometers is given by

$$P(h) = 1035e^{-0.12h} \tag{8}$$

Find **(a)** the atmospheric pressure at 40 km, and **(b)** the altitude where the atmospheric pressure drops to 20 percent of that at sea level.

Solution

(a) From Equation (8), $P(40) = 1035e^{(-0.12)(40)} \approx 8.5$. Hence the atmospheric pressure at 40 km ($\approx$25 mi) is only 8.5 g/cm^2, less than 1 percent of the pressure at sea level.

(b) Find the value of h for which $P(h)$ is 20 percent of the pressure at sea level. Replace $P(h)$ in Equation (8) by (0.2)(1035) and solve for h.

$$(0.2)(1035) = 1035e^{-0.12h}, \qquad e^{-0.12h} = 0.2,$$
$$-0.12h = \ln 0.2, \qquad h = \frac{\ln 0.2}{-0.12} \approx 13.4.$$

Since $h \approx 13.4$, at an altitude of 13.4 km ($\approx$8.3 mi, not quite 44,000 ft), the atmospheric pressure drops to 20 percent of the atmospheric pressure at sea level. ◄

In the previous section we saw an example of an application of logarithms to measure sound levels. The next two examples also illustrate models that apply logarithms.

Measuring Earthquakes

An earthquake produces seismic waves whose amplitude is measured on a seismograph. Charles Richter, an American geologist, recognized the great variation in amplitudes of earthquakes and proposed a logarithmic scale to measure their severity. The magnitude $M(A)$ of an earthquake with amplitude A is a number on the Richter scale given by

$$M(A) = \log\left(\frac{A}{A_0}\right), \tag{9}$$

where A_0 is a standard amplitude.

►EXAMPLE 7 ***Comparing earthquakes*** How many times larger was the amplitude of the Alaskan earthquake on March 28, 1964, which measured 8.6 on the Richter scale, than the amplitude of a relatively minor aftershock that measured 4.3?

Strategy: Use Equation (9). Let A_1 and A_2 be the two amplitudes, and replace $M(A_1)$, $M(A_2)$, by 8.6 and 4.3, respectively. Find A_1 and A_2 in terms of A_0.

Solution

Follow the strategy.

$$8.6 = \log\left(\frac{A_1}{A_0}\right) \quad \text{and} \quad 4.3 = \log\left(\frac{A_2}{A_0}\right).$$

Write each equation in exponential form and solve for A_1 and A_2.

$$A_1 = A_0 10^{8.6} \quad \text{and} \quad A_2 = A_0 10^{4.3}.$$

Solve the second equation for A_0 and substitute into the first equation,

$$A_0 = A_2 10^{-4.3}, \text{ so } A_1 = (A_2 10^{-4.3})10^{8.6} = 10^{4.3}A_2 \approx 19{,}953A_2.$$

The amplitude A_1 of the 8.3 magnitude earthquake is nearly 20,000 times larger than the amplitude of the 4.3 aftershock, which explains the enormous amount of damage done by the original earthquake. ◄

Acidity Measurement

Chemists determine the acidity of a solution by measuring the hydrogen ion concentration (denoted by $[H^+]$, in moles per liter). Such concentrations are very small numbers. To deal with numbers in a more familiar range, the quantity denoted by pH essentially puts hydrogen ion concentration on a logarithmic scale.

Formula for determining acidity of a solution

For a solution with hydrogen ion concentration of $[H^+]$ moles per liter, the corresponding pH value is given by

$$\text{pH} = -\log[H^+]. \quad \textbf{(10)}$$

If the pH number for a solution is less than 7, then the solution is called *acidic;* if the pH is greater than 7, then the solution is called *basic.* Solutions with pH equal to 7 are called *neutral.* For a solution with 10^{-1} moles of hydrogen ions per liter ($[H^+] = 10^{-1}$), the pH is $-\log 10^{-1} = -(-)1 = 1$; such a solution is very strongly acidic (even one-tenth of a mole of hydrogen ions indicates *lots* of freely reacting ions in the solution). At the other end of the scale, if $[H^+] = 10^{-13}$, then $\text{pH} = -\log(10^{-13}) = 13$, indicating a strongly basic solution.

►EXAMPLE 8 ***Fruit juice acidity*** A certain fruit juice has a hydrogen ion concentration of 3.2×10^{-4} moles per liter. Find the pH value for the juice and decide whether it is acidic or basic.

Solution

Given that $[H^+] = 3.2 \times 10^{-4}$, substitute into Equation (10):

$$\text{pH} = -\log(3.2 \times 10^{-4}) = -\log(0.00032) \approx 3.5.$$

A pH of less than 7 indicates that the juice would be classified as acidic. ◄

EXERCISES 4.5

Check Your Understanding

Exercises 1–6 If \$1000 is invested in an account that earns interest compounded continuously at an interest rate that doubles the investment in value every 12 years, then select from the choices below the amount that is closest to the total value of the investment after the indicated period of time. As in the text $A(t)$ denotes the amount of money in the account t years after the investment is made.

(a) \$1400 **(b)** \$1500 **(c)** \$2000
(d) \$2800 **(e)** \$3000 **(f)** \$4000
(g) \$6000 **(h)** \$7000 **(i)** \$8000

1. $A(24) =$ ______.
2. $A(36) =$ ______.
3. $A(6) =$ ______.
4. $A(18) =$ ______.
5. The interest earned during the first 18 years is ______.
6. The interest earned during the years from $t = 12$ to $t = 24$ is ______.

Exercises 7–10 A radioactive substance has a half-life of 30 days. Select from the list below the choice that is closest to the amount of the substance that remains after the indicated period of time. A_0 denotes the number of grams of the substance when t is 0, and $A(t)$ denotes the number of grams t days later.

(a) $0.25A_0$ **(b)** $0.35A_0$ **(c)** $0.50A_0$
(d) $0.70A_0$ **(e)** $0.75A_0$ **(f)** $0.80A_0$

7. $A(60) =$ ______.
8. $A(15) =$ ______.
9. $A(45) =$ ______.
10. The amount of the substance that decays during the first 60 days is ______.

Develop Mastery

Exercises 1–2 ***Number of Bacteria***

1. The number of bacteria in a culture doubles every 1.5 hours. If 4000 are present initially,
   (a) How many will there be three hours later?
   (b) Four hours later?
   (c) How long does it take for the number to increase to 40,000?
2. If the number of bacteria in a sample increases from 1000 to 1500 in two hours, how long does it take for the number of bacteria
   (a) to double? **(b)** to triple?

Exercises 3–5 ***Population Growth***

3. The world population in 1968 was 3.5 billion; in 1992 it was 5.5 billion. Assume exponential growth.
   (a) Predict the population in the year 2000.
   (b) When will the population reach 7 billion?

4. Assuming an annual population increase of 1.5 percent since 1968 (when the world population was 3.5 billion)
(a) Show that n years after 1968, the population $P(n)$ is $(3.5)(1.015)^n$ billion.
(b) Determine the population for 1992. How does your calculation agree with the information in Exercise 3?

5. In 1960 the population of the United States was 180 million; in 1970 it was 200 million. Assume an exponential rate of growth and predict the population for the year 2000.

Exercises 6–12 ***Compound Interest*** *Assume interest is compounded continuously and that all interest rates are annual.*

6. Suppose \$1000 is invested in an account that earns 8 percent interest.
(a) How much interest is in the account 10 years later?
(b) How long does it take the money to double?
(c) To triple?

7. Suppose \$1000 invested in a savings account increases over three years to \$1200. What rate of interest is being paid?

8. An investment of \$800 in a savings certificate that pays 10 percent interest has grown to \$2000. How many years ago was the certificate purchased?

9. An investment doubles in 8 years. What is the rate of interest?

10. How long does it take for an investment to double if the rate of interest is
(a) 8 percent? **(b)** 12 percent? **(c)** r percent?

11. Suppose you invest \$1000 in a savings account at 5 percent interest, and at the end of 6 years you use the accumulated total to purchase a savings certificate that earns 6 percent interest. What is the value of the savings certificate 6 years later?

12. An annuity pays 12 percent interest. What amount of money deposited today will yield \$3000 in 8 years?

Exercises 13–18 ***Radioactive Decay***

13. A radioactive isotope, radium-226, has a half-life of 1620 years. A sample contained 10 grams in 1900. How many grams will remain in the year
(a) 2000? **(b)** 3000?

14. Radioactive lead, lead-212, has a half-life of 11 days. How long will it take for 20 pounds of lead-212 to decay to 8 pounds?

15. Another isotope of lead, lead-210, has a half-life of 22 years. How much of a 10-pound sample would remain after 10 years?

16. Radioactive iodine-131 is a component of nuclear fallout.
(a) If 10 mg of iodine-131 decays to 8.4 mg in 2 days, what is the half-life of the isotope?
(b) In how many days does the 10 mg sample decay to 2 mg?

17. After two years, a sample of a radioactive isotope has decayed to 70 percent of the original amount. What is the half-life of the isotope?

18. A 12 mg sample of radioactive polonium decays to 7.26 mg in 100 days.
(a) What is polonium's half-life?
(b) How much of the 12 mg sample remains after six months (180 days)?

Exercises 19–22 ***Carbon Dating*** *Use the carbon dating information discussed in this section.*

19. A piece of petrified wood contains 40 percent of its original amount of C^{14}. How old is it?

20. If the Dead Sea Scrolls contain about 80 percent of their original C^{14}, how old are they?

21. How old is a fossil skeleton that contains 85 percent as much C^{14} as a living person?

22. If the Piltdown cranium (the Historical Note, "Exponential Functions, Dating, and Fraud Detection") was found to contain 93 percent of the C^{14} found in a modern skeleton, what is the approximate age of the cranium?

Exercises 23–25 ***Earthquake Comparisons***

23. A 1933 earthquake in Japan registered 8.9 on the Richter scale, the highest reading ever recorded. Compare its amplitude to that of the 1971 earthquake in San Fernando, California, which measured 6.5.

24. The famous San Francisco earthquake of 1906 registered 8.4 on the Richter scale. Compare its amplitude with that of the 1976 earthquake in Guatemala, which measured 7.9.

25. If an earthquake in Ethiopia had an amplitude 100 times larger than an earthquake that measured 5.7 on the Richter scale, what would the Ethiopian earthquake measure?

Exercises 26–27 ***Atmospheric Pressure***

26. Example 6 contains a formula for the atmospheric pressure $P(h)$ (in g/cm^2) at an altitude of h km. If h is measured in miles and pressure is measured in lb/in^2, then the corresponding equation is $P(h) = 14.7e^{-0.19h}$.
(a) Find the atmospheric pressure at an altitude of 25 miles.
(b) At what altitude is the atmospheric pressure one-tenth of that at sea level?

27. Use the equation for atmospheric pressure in Example 6 to find the altitude at which the atmospheric pressure is 100 g/cm^2.

28. A satellite is powered by a radioactive isotope. The power output $P(t)$ (measured in watts) generated in t days is given by $P(t) = 50e^{-t/250}$.
(a) How much power is available at the end of a year (365 days)?
(b) What is the half-life of the power supply?
(c) If the equipment aboard the satellite requires 10 watts of power to operate, what is the operational life of the satellite?

Exercises 29–30 ***Acidity***

29. The hydrogen ion concentration for a sample of human blood is found to be 4.5×10^{-8} moles per liter. Find the pH value of the sample. Is it acidic or basic?

30. Find the pH value for
(a) vinegar, $[H^+] = 6.3 \times 10^{-4}$
(b) milk, $[H^+] = 4 \times 10^{-7}$
(c) water, $[H^+] = 5.0 \times 10^{-8}$
(d) sulphuric acid, $[H^+] = 1$.

31. Oil is being pumped from a well. If we assume that production is proportional to the amount of oil left in the well, then it can be shown that the number of barrels of oil, $A(t)$, left in the well t years after pumping starts, is given by $A(t) = Ce^{-kt}$, where C and k are constants. When t is 0 it is estimated that the well holds 1 million barrels of oil, and after six years of pumping, 0.5 million barrels remain. It is not profitable to keep pumping when fewer than 50,000 barrels remain in the well. What is the total number of years during which pumping remains profitable?

32. The population of Taunton is growing exponentially at an annual rate of 5 percent.
(a) Show that after t years the population increases from 13,000 to N (in thousands) given by $N = 13(1.05)^t$.
(b) In how many years will the population double?
(c) In how many years will the population triple?

33. ***Looking Ahead to Calculus*** A 500 gallon tank of brine starts the day with 150 pounds of salt. Fresh water runs into the tank at the rate of 5 gallons per minute and the well-stirred mixture drains at the same rate. In calculus it can be shown that the number of pounds, $A(t)$, of salt still in the tank t minutes later is given by $A(t) = 150e^{-0.01t}$.
(a) How many pounds of salt remain in the tank after 30 minutes?
(b) How many minutes does it take to reduce the amount of salt in the tank to 50 pounds?

34. In Example 3 we developed an equation for the amount $A(t)$ of strontium-90 (half-life 29 years) left after t years, starting with an initial amount A_0: $A(t) = A_0e^{-0.0239t}$. Show that $A(t)$ is also given by $A(t) = A_0(2^{-t/29})$. (*Hint:* In Example 3 $k = \frac{\ln 2}{29}$, so $A(t) = A_0e^{-(t \ln 2)/29}$.)

35. ***Spreading a Rumor*** A rumor is spreading about the safety of county drinking water. Suppose P people live in the county and $N(t)$ is the number of people who have not yet heard the rumor after t days. If the rate at which $N(t)$ decreases is proportional to the number of people who have not yet heard the rumor, then $N(t)$ is given by $N(t) = Pe^{-kt}$, where k is a constant to be determined from observed information. In Calaveras County, population 50,000, suppose 2000 people have heard the rumor after the first day (when t is 1).
(a) How many people will have heard the rumor after 10 days?
(b) After how many days will half of the population have heard the rumor?

36. Use the equation in Exercise 35. If 10 percent of a county population of 20,000 have heard the rumor after the first two days, then how many people will have heard the rumor after three additional days?

37. ***Hatchee Reservoir Contamination*** In Example 4 suppose the pollutant level decreases by one half every 10 days. What percentage of the toxic chemical will be present on
(a) May 21? **(b)** July 4?

38. Draw a calculator graph of $y = (1 + x)^{1/x}$. Use the graph to see what happens to y when $x \to 0$.

CHAPTER 4 REVIEW

Test Your Understanding

True or False. Give reasons.

1. $\ln x$ is positive for every positive x.

2. $\ln x^2$ is defined for every real number x.

3. $\ln 1 = e$.

4. $\ln e = 1$.

5. $(\ln x)^2 = 2(\ln x)$ for every $x > 0$.

6. If x and y are positive numbers, then $\ln(\frac{x}{y}) = \frac{\ln x}{\ln y}$.

7. $\frac{1}{2}(\ln x^2) = \ln x$ for every positive x.