Math 1100 Final Name: Solutions

Created: Fri Apr 07 12:03:04 MDT 2006

Directions:Work all problems included. If you need more room use the back of the page to complete the problem. You may a calculator for computing numerical values. However, you may not use calculators for symbolic calculations unless the problem indicates this can be done. For full credit you must show all work including algebraic steps.

Problem 1: Use integration by parts to compute the following defintie integral.

$$\int_0^1 t\ln(2t+1)\,dt$$

$$\int t\ln(2t+1)\,dt$$

$u = \ln(2t+1)$ $\quad dv = t$

$du = \frac{2}{2t+1}\,dt$ $\quad v = \frac{1}{2}t^2$

$$= \frac{1}{2}t^2\ln(2t+1)\Big|_0^1 - \int_0^1 \frac{1}{2}t^2 \cdot \frac{2}{2t+1}\,dt$$

$$= \left[\frac{1}{2}(1)^2\ln(3) - \frac{1}{2}(0)^2\ln(1)\right] - \int_0^1 \frac{t^2}{2t+1}\,dt$$

$$= \frac{1}{2}\ln(3) - \int_0^1 \frac{t^2}{2t+1}\,dt$$

$u = 2t+1$

$du = 2dt$

$t^2 = \left(\frac{1}{2}(u-1)\right)^2$

$t=0 \Rightarrow u=1$

$t=1 \Rightarrow u=3$

$$= \frac{1}{2}\ln(3) - \int_1^3 \frac{\frac{1}{4}(u-1)^2}{u}\cdot\frac{1}{2}du$$

$$= \frac{1}{2}\ln(3) - \frac{1}{8}\int_1^3 \frac{u^2-2u+1}{u}\,du$$

$$= \frac{1}{2}\ln(3) - \frac{1}{8}\int_1^3 \left(u - 2 + \frac{1}{u}\right)du$$

$$= \frac{1}{2}\ln(3) - \frac{1}{8}\left(\frac{1}{2}u^2 - 2u + \ln|u|\right)$$

over

Problem 2: Suppose we want to analyze the function $f(x) = 3x^4 - 8x^3 + 6x^2$. **a.** Determine all points where the concavity of the function is zero. **b.** Which of these points are inflection points.?

$f'(x) = 12x^3 - 24x^2 + 12x$

$f''(x) = 36x^2 - 48x + 12$

$= 12(3x^2 - 4x + 1)$

$= 0$

$f''(0) = 12 > 0$ | $f''(2/3) = -4 < 0$ | $f''(2) = 60 > 0$

$\cup$ up $\quad \frac{1}{3} \quad$ down $\quad 1 \quad$ up

Set $3x^2 - 4x + 1 = 0$

$$\Rightarrow \begin{cases} x = \frac{4 \pm \sqrt{16-12}}{6} \\ \quad = \frac{4 \pm \sqrt{4}}{6} \end{cases} \Rightarrow$$

$x_1 = \frac{4+2}{6} = 1$

$x_2 = \frac{4-2}{6} = \frac{2}{6} = \frac{1}{3}$

Since the concavity changes at both points, $x = \frac{1}{3}$, and $x=1$ are both inflection pts

$$= \frac{1}{2}\ln(3) - \frac{1}{8}\left(\frac{1}{2}u^2 - 2u + \ln(u)\right)\Big|_{u=1}^{u=3}$$

$$= \frac{1}{2}\ln(3) - \frac{1}{8}\left[\left(\frac{1}{2}(3)^2 - 2(3) + \ln(3)\right) - \left(\frac{1}{2}(1)^2 - 2(1) + \ln(1)\right)\right]$$

This is far enough. You can use a calculator to find the numerical value

Note: This is a tough problem and requires practice.

Problem 3: Compute the derivative of the following function using the quotient rule.

$$g(x) = \frac{2x^2 - x}{7 - 3x}$$

$$g'(x) = \frac{\left(\frac{d}{dx}(2x^2 - x)\right) \cdot (7-3x) - (2x^2 - x)\frac{d}{dx}(7-3x)}{(7-3x)^2}$$

$$= \frac{(4x-1)(7-3x) - (2x^2-x)(-3)}{(7-3x)^2}$$

Problem 4: Compute the following indefinite integral

$$\int_1^2 \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}\, dx$$

Let: $u = \sqrt{x+1} = (x+1)^{1/2}$

$du = \frac{1}{2}(x+1)^{-1/2}\, dx \Rightarrow 2\, du = (x+1)^{-1/2}\, dx = \frac{1}{\sqrt{x+1}}\, dx$

$x = 1 \Rightarrow u = \sqrt{2}$

$x = 2 \Rightarrow u = \sqrt{3}$

Then:

$$\int_1^2 \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}\, dx = 2\int_{\sqrt{2}}^{\sqrt{3}} e^u\, du = 2e^u \Big|_{u=\sqrt{2}}^{u=\sqrt{3}}$$

$$= 2e^{\sqrt{3}} - 2e^{\sqrt{3}}$$

Problem 5: Suppose that the you know the following limits at $x = -2$

$$\lim_{x\to-2} f(x) = 0, \quad \lim_{x\to-2} g(x) = 2, \quad \text{and} \lim_{x\to-2} h(x) = -3,$$

compute the following limits.

$\lim_{x\to-2}[2h(x) - g(x)] =$ $2(-3) - 2 = -6 - 2 = -8$

$\lim_{x\to-2}(2 + g(x)) =$ $2 + 2 = 4$

$\lim_{x\to-2} f(x)g(x) =$ $0 \cdot 2 = 0$

Problem 6: Implicit Differentiation: Compute $\frac{dy}{dx}$ using the following implicit relationship

$$xy^2 + e^{xy} = y2^x$$

Note that you can leave $\frac{dy}{dx}$ in terms of an expression in x and y, but you must solve for $\frac{dy}{dx}$.

$\frac{d}{dx}(xy^2+e^{xy}) = \frac{d}{dx}(y2^x)$

$\Rightarrow y^2 + 2xy\frac{dy}{dx} + e^{xy}(y + x\frac{dy}{dx}) = \ln(2)2^x y + 2^x \frac{dy}{dx}$

$\Rightarrow (2xy + e^{xy} \cdot x - 2^x)\frac{dy}{dx} = \ln(2)2^x \cdot y - y^2 - ye^{xy}$

$\Rightarrow \frac{dy}{dx} = \frac{\ln(2)2^x y - y^2 - ye^{xy}}{2xy + xe^{xy} - 2^x}$

Problem 7: For the function

$$h(t) = 0.6t^2 - 2.7t + 9.3$$

find the instantaneous rate of change of the function at $t = 1.6$. Compare the instantaneous rate of change to the average rate of change on the interval $[1.55, 1.65]$.

$h'(t) = 1.2t - 2.7$

$h'(1.65) = 1.2(1.6) - 2.7$

$= -0.78$

instantaneous

$h(1.65) = 6.4785$

$h(1.55) = 6.5565$

The average rate of change is:

$$\frac{h(1.65) - h(1.55)}{1.65 - 1.55} = \frac{6.4785 - 6.5565}{0.1} = -0.78$$

They are the same in this case.

Problem 8: Find all critical points of the function

$$f(x, y) = x^2 + 4xy - y^2 - 3x - 2y + 17$$

Classify the critical point as a relative minimum, relative maximum, or saddle point using the second derivative test.

$\frac{\partial f}{\partial x} = 2x + 4y - 3 = 0$

$\frac{\partial f}{\partial y} = 4x - 2y - 2 = 0$

$\Rightarrow \begin{cases} 2x + 4y = 3 \\ 4x - 2y = 2 \end{cases}$

$\Rightarrow 10x = 7 \Rightarrow x = 7/10$ (add 2*eq 2 to eq 1)

$10y = 4 \Rightarrow y = 4/10 = 2/5$

The only critical point is

$(x, y) = (7/10, 4/10)$

$d = \frac{\partial^2 f}{\partial x} \cdot \frac{\partial^2 f}{\partial y} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$

$\frac{\partial^2 f}{\partial x} = 2$

$\frac{\partial^2 f}{\partial y^2} = -2$

$\frac{\partial^2 f}{\partial x \partial y} = 4$

$\Rightarrow d = (2)(-2) - (4)^2$

$= -4 - 16 = -20 < 0$

$\Rightarrow$ The critical point is a saddle point.

Problem 9: Compute the general solution of the following equation via separation of variables.

$$\frac{dy}{dx} = \frac{x^2+3}{y^2-1}$$

Leave the solution as an implicit relationship for x and y.

$$(y^2-1)\frac{dy}{dx} = x^2+3$$

$$\Rightarrow (y^2-1)\underbrace{\frac{dy}{dx}dx}_{dy} = (x^2+3)\,dx$$

$$\Rightarrow (y^2-1)\,dy = (x^2+3)\,dx$$

$$\Rightarrow \int (y^2-1)\,dy = \int (x^2+3)\,dx$$

$$\Rightarrow \frac{1}{3}y^3 - y = \frac{1}{3}x^3 + 3x + C$$

Problem 10: Compute the derivative of the following function using the chain rule.

$$h(x) = 2(x^2+3x+7)^4$$

$$h'(x) = 2(4)(x^2+3x+7)^3 \cdot \frac{d}{dx}(x^2+3x+7)$$

$$= 8(x^2+3x+7)^3(2x+3)$$