Math 1100 Test 3 Name: Solutions

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**Directions:Work all problems included. If you need more room use the back of the page to complete the problem. You may a calculator for computing numerical values. However, you may not use calculators for symbolic calculations unless the problem indicates this can be done. For full credit you must show all work including algebraic steps.**

**Problem 1:** Compute the following indefinite integral

$$\int_1^2 \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}\,dx \longrightarrow = \int_{\sqrt{2}}^{\sqrt{3}} e^u\,(2\,du)$$

$u = \sqrt{x+1}$

$x = 1, \quad u = \sqrt{1+1} = \sqrt{2}$

$x = 2, \quad u = \sqrt{2+1} = \sqrt{3}$

$du = \frac{1}{2\sqrt{x+1}}\,dx$

$\Rightarrow 2\,du = \frac{1}{\sqrt{x+1}}$

$$= 2\int_{\sqrt{2}}^{\sqrt{3}} e^u\,du$$

$$= 2e^u \Big|_{u=\sqrt{2}}^{u=\sqrt{3}}$$

$$= 2e^{\sqrt{3}} - 2e^{\sqrt{2}}$$

$$= 2\left(e^{\sqrt{3}} - e^{\sqrt{2}}\right)$$

**Problem 2:** Verify that the function is a solution of the differential equation.

$y = 10e^{-3t} \qquad y' + 3y = 0$

$$y' = 10\left(-3e^{-3t}\right) = -30e^{-3t}$$

$$\Rightarrow y' + 3y = -30e^{-3t} + 3\left(10e^{-3t}\right)$$

$$= -30e^{-3t} + 30e^{-3t}$$

$$= 0 \ \checkmark$$

**Problem 3:** Decide whether the variables in the differential equation can be separated.

$$x\frac{dy}{dx} = \frac{x^2}{y^2}$$

$\Rightarrow \quad y^2 \frac{dy}{dx} = \frac{x^2}{x}$

$\Rightarrow \quad y^2 \frac{dy}{dx} \cdot dx = x\,dx$

$\Rightarrow \quad \underbrace{y^2 dy}_{\text{depends only on } y} = \underbrace{x\,dx}_{\text{depends only on } x}$ $\Rightarrow$ the equation is separable

**Problem 4:** Find all critical points of the function

$$f(x,y) = x^3 + y^3 - 4x - 9y + 17$$

Classify the critical point as a relative minimum, relative maximum, or saddle point using the second derivative test.

$\frac{\partial f}{\partial x} = 3x^2 - 4 = 0 \Rightarrow x^2 = 4/3 \Rightarrow x = \pm\sqrt{4/3}$

$\frac{\partial f}{\partial y} = 3y^2 - 9 = 0 \Rightarrow y^2 = 3 \Rightarrow x = \pm\sqrt{3}$

$\Rightarrow$ 4 critical pts $(\sqrt{4/3}, \sqrt{3})$, $(\sqrt{4/3}, -\sqrt{3})$, $(-\sqrt{4/3}, \sqrt{3})$, $(-\sqrt{4/3}, -\sqrt{3})$

Now the second derivatives

$\frac{\partial^2 f}{\partial x^2} = 6x$

$\frac{\partial^2 f}{\partial y^2} = 6y$

$\frac{\partial^2 f}{\partial x \partial y} = 0 = \frac{\partial^2 f}{\partial y \partial x}$

$\Rightarrow d = (6x)(6y) = 36xy$

① $(\sqrt{4/3}, \sqrt{3})$ $\quad d = 36\sqrt{4/3}\sqrt{3} > 0$ $\Rightarrow$ rel min. since $\partial^2 f/\partial x^2 > 0$

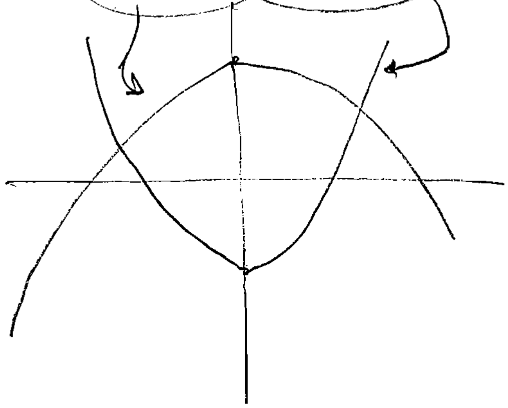
② $(\sqrt{4/3}, -\sqrt{3})$ $\quad d = 36\sqrt{4/3}(-\sqrt{3}) < 0$ $\Rightarrow$ saddle pt

③ $(-\sqrt{4/3}, \sqrt{3})$ $\quad d = 36(-\sqrt{4/3})\sqrt{3} < 0$ $\Rightarrow$ saddle pt.

④ $(-\sqrt{4/3}, -\sqrt{3})$ $\quad d = 36(-\sqrt{4/3})(-\sqrt{3}) > 0$ $\Rightarrow$ rel max since $\partial^2 f/\partial x^2 < 0$

**Problem 5:** Sketch the region between the graphs of the functions and compute the area of this region.

$y = 9 - x^2$, $y = x^2 - 4$



Find intersection points:

$9 - x^2 = x^2 - 4 \Rightarrow 2x^2 = 13$

$\Rightarrow x = \pm\sqrt{13/2}$

$$\text{Area} = \int_{-\sqrt{13/2}}^{+\sqrt{13/2}} \left((9-x^2) - (x^2-4)\right) dx$$

$$= \int_{-\sqrt{13/2}}^{\sqrt{13/2}} (13 - x^2)\, dx$$

$$= \left(13x - \frac{1}{3}x^3\right)\Big|_{-\sqrt{13/2}}^{\sqrt{13/2}} = \text{Area}$$

**Problem 6:** Verify that the general solution satisfies the differential equation. Then find the particular solution that satisfies the initial condition.

| | |
|---|---|
| General Solution: | $y = (1/3)x^2 + C$ |
| Differential Equation: | $3y' = 2x$ |
| Initial Condition: | $y = 3$ and $x = 2$ |

$y' = \frac{d}{dx}\left(\frac{1}{3}x^2 + c\right) = \frac{2}{3}x + 0$

$3y' = 3\left(\frac{2}{3}x\right) = 2x$

same $\Rightarrow$ This is a solution

When $y = 3$ & $x = 2$

$\Rightarrow \quad 3 = \left(\frac{1}{3}\right)(2)^2 + C$

$\Rightarrow \quad 3 = \frac{4}{3} + C \Rightarrow C = 3 - \frac{4}{3} = \frac{5}{3}$

$y = \frac{1}{3}x^2 + \frac{5}{3}$

**Problem 7:** Evaluate $g_x$ and $g_y$ at the point.

$$g(x,y) = x^2x^2 - 6y - 4x + 10 \qquad (1,-1)$$

$\frac{\partial g}{\partial x} = g_x = 4x^3 - 4$ at $x = 1, y = -1$

$g_x = 4(1)^3 - 4 = 0$

$\frac{\partial g}{\partial y} = g_y = -6$ at $x = 1, y = -1$

$=$ constant value

$g_y = -6$

**Problem 8:** Compute the general solution of the following equation via separation of variables.

$$\frac{dy}{dx} = \frac{x^2+3}{y^2-1}$$

Leave the solution in an implicit relationship for $x$ and $y$.

$\Rightarrow (y^2-1)\frac{dy}{dx} = x^2+3$

$\Rightarrow (y^2-1)\underbrace{\frac{dy}{dx}dx}_{dy} = (x^2+3)dx$

$\Rightarrow (y^2-1)dy = (x^2+3)dx$

$\Rightarrow \int(y^2-1)dy = \int(x^2+3)dx$

$\Rightarrow \frac{1}{3}y^3 - y = \frac{1}{3}x^3 + 3x + C$

Should be $y^2$ not $y^3$

**Problem 9:** Find all critical points of the function

$$f(x,y) = -x^2 - 9xy - y^3 - 6x - 8y$$

Classify the critical point as a relative minimum, relative maximum, or saddle point using the second derivative test.

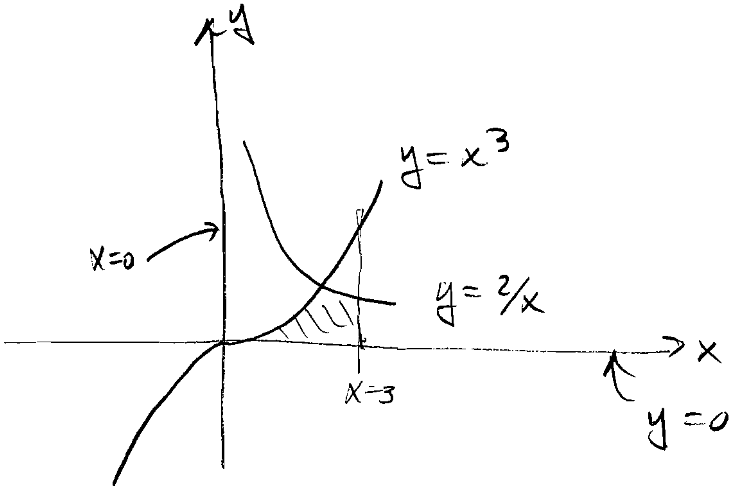
$$\frac{\partial f}{\partial x} = -2x - 9y - 6$$
$$\frac{\partial f}{\partial y} = -9x - 3y^2$$

This problem has a typo

leave this one out for now

**Problem 10:** Sketch the region between the graphs of the functions and compute the area of this region.

$$y = \frac{2}{x}, \quad y = x^3, \quad y = 0, \quad x = 0, \quad x = 3$$




Intersection Point

$x^3 = 2/x$

$\Rightarrow x^4 = 2$

$\Rightarrow x = 2^{1/4}$

So,

$$\text{Area} = \int_0^{\sqrt[4]{2}} x^3\,dx + \int_{\sqrt[4]{2}}^{3} 2/x\,dx$$
$$= \frac{1}{4}x^4 \Big|_0^{\sqrt[4]{2}} + 2\ln(x)\Big|_{\sqrt[4]{2}}^{3}$$
$$= \frac{1}{4}(2) - 0 + 2\ln(3) - 2\ln(2^{1/4})$$
$$= \frac{1}{2} + 2\ln(3) - \frac{1}{8}\ln(2)$$