**Problem Definition**

Problem 23. Use separation of variables to find the general solution of the differential equation.

$$y' = \frac{x}{y} - \frac{x}{1+y}$$

**Solution Step 1:**

We can start by doing some algebra on the right hand side of the differential equation. The differential equation can be written as

$$\begin{aligned} y' &= \frac{x}{y} - \frac{x}{1+y} \\ &= \frac{x(1+y) - xy}{y(1+y)} \\ &= \frac{x + xy - xy}{y(1+y)} \\ &= \frac{x}{y(1+y)} \end{aligned}$$

To separate variables, we multipley both sides by $dx$ to obtain.

$$y'\ dx = \frac{x}{y(1+y)}\ dx$$

Clearing functions by multiplying the denominator on the right hand side gives

$$y(1+y)\ dy = x\ dx$$

So this equation is separable. The left hand side depends only on the variable $y$ and the right hand side depends only on the variable $x$.

**Solution Step 2:**

The next step is to integrate each side of the transformed equation. The left hand side is

$$\int y(1=y)\ dy = \int (y - y^2)\ dy = \frac{y^2}{2} - \frac{y^3}{3} + C_1$$

and

$$\int x \, dx = \frac{x^2}{2} + C_2$$

Setting the integrals equal we find

$$\frac{y^2}{2} - \frac{y^3}{3} + C_1 = \frac{x^2}{2} + C_2$$

or

$$\frac{y^2}{2} - \frac{y^3}{3} = \frac{x^2}{2} + C$$

where $C = C_2 - C_1$. This defines the solution in terms of an implicit relationship between $x$ and $y$. Note that it is unlikely to solve for $y$ in terms of $x$ in this case.