Problem Definition

Problem 41. **Investment** A brokerage firm opens a new real estate investment plan for which the earnings are equivalent to continuous compounding at the rate of r. The firm estimates that deposits from investors wil create a net cash flow of Pt dollars, where t is the time in years. The rate of change in the total investment A is modeled by

$$\frac{dA}{dt} = rA + Pt$$

(a) Solve the differential equation and find the total investment A as a function of time, t. Assume that $A = 0$ when $t = 0$.

(b) Find thetotal investment A after 10 years given that $P = \$500,000$ and $r = 9\%$.

Solution Step 1:

The first step is to rewrite the differential equation in the form of a first order linear differential equation. That is, rewrite the equation as

$$\frac{dA}{dt} - rA = Pt$$

In terms of the definition of a first order linear equation we can identify

$$P(t) = -r$$

and

$$Q(t) = Pt$$

It is unfortunate notation, but the $P(t)$ in the integrating factor above is not the same as the P in the differential equation. However, if you keep that in mind we should be ok.

Solution Step 2:

The integrating factor is defined by

$$u(t) = e^{\int -r dt} = e^{-rt}$$

Solution Step 3:

With the integrating factor computed, the solution is

$$\begin{aligned} A(t) &= \frac{1}{u(t)} \int Ptu(t)dt \\ &= \frac{1}{e^{-rt}} \int Pte^{-rt}dt \\ &= Pe^{rt} \int te^{-rt}dt \end{aligned}$$

At this point we need to compute the integral via an integration by parts.

Solution Step 4:

To compute the integral necessary for the solution we will need to apply our knowledge of integration by parts. Use $u = t$ and $dv = e^{-rt}dt$.

$$\begin{aligned} \int te^{-rt}dt &= t\left(\frac{1}{-r}e^{-rt}\right) - \int \left(-\frac{1}{r}e^{-rt}\right) dt \\ &= -\frac{t}{r}e^{-rt} + \frac{1}{r}\int e^{-rt}dt \\ &= -\frac{t}{r}e^{-rt} - \frac{1}{r^2}e^{-rt} \\ &= -\frac{e^{-rt}}{r^2}(1+rt) \end{aligned}$$

Solution Step 5:

The solution is of the form

$$\begin{aligned} A(t) &= Pe^{rt}\left(-\frac{e^{-rt}}{r^2}(1+rt) + C\right) \\ &= P\left(-\frac{1}{r^2}(1+rt) + Ce^{rt}\right) \end{aligned}$$

Solution Step 6:

The initial value that we have to apply is

$$\begin{aligned} A(0) &= P\left(-\frac{1}{r^2}(1+r(0)) + Ce^{r(0)}\right) \\ &= P\left(-\frac{1}{r^2} + C\right) = 0 \end{aligned}$$

Then

$$P\left(\frac{1}{r^2} + C\right) = 0$$

which implies $C = r^{-2}$. This means the solution can be written as

$$\begin{aligned} A(t) &= \frac{P}{r^2}\left(-(1+rt) + e^{rt}\right) \\ &= \frac{P}{r^2}\left(e^{rt} - (1+rt)\right) \end{aligned}$$

Solution Step 7:

The last step is to approximate the amount in the investment given when $P = \$500,000$ and $r = 9\%$. In this case,

$$A(t) = \frac{500000}{(0.09)^2}\left(e^{(0.09)t} - 1 - (0.09)t\right)$$

and

$$A(10) = \frac{500000}{(0.09)^2}\left(e^{(0.09)(10)} - 1 - (0.09)(10)\right) \approx 34,543,402$$