

Problem Definition

Problem 15. **Biology** At any time t , the rate of growth of the population N of deer in a state park is proportional to the product of N and $L - N$, where $L = 500$ is the maximum number of deer the park can maintain. When $t = 0$, $N = 100$ and when $t = 4$, $N = 200$. Write N as a function of t .

Solution Step 1:

The first step in this problem is to build a differential equation that models the process described in the problem. The rate of growth of a deer herd is proportional to the number in the herd and the difference between the maximum number of deer and the current number of deer. The proportionality relationship is the following.

$$\frac{dN}{dt} \propto N(L - N)$$

where the symbol \propto is the proportionality symbol. To finish off the equation we can define the proportionality constant to be k and write

$$\frac{dN}{dt} = kN(500 - N)$$

given the limiting population number of $L = 500$. Also, the initial population is $N(0) = 100$. This model is referred to as the logistic model of population growth.

Solution Step 2:

We can use separation for variables to compute the solution. The separation of variables will produce an equation of the form

$$\frac{dN}{N(500 - N)} = k dt$$

The left hand side can be integrated by partial fractions. The integral on the left hand side of the equation is computed as follows

$$\int \frac{dN}{N(500 - N)} = \int \left(\frac{1/500}{N} + \frac{1/500}{500 - N} \right)$$

$$\begin{aligned}
&= \frac{1}{500} \left(\int \frac{1}{N} dN + \int \frac{1}{500 - N} dN \right) \\
&= \frac{1}{500} (\ln|N| - \ln|500 - N|) + C_1 \\
&= \frac{1}{500} \left(\ln \left| \frac{N}{500 - N} \right| \right) + C_1
\end{aligned}$$

This result can be substituted back into the separation of variables equation to find

$$\frac{1}{500} \left(\ln \left| \frac{N}{500 - N} \right| \right) + C_1 = \int k dt = kt + C_2$$

Multiplying the equation by 500 gives

$$\ln \left| \frac{N}{500 - N} \right| = 500kt + 500(C_2 - C_1) = 500kt + C_3$$

Exponentiating both sides of this results gives

$$\frac{N}{500 - N} = e^{500kt + C_3} = C e^{500kt}$$

Solution Step 4:

The next step is to solve for N and compute the rate constant and C . One thing we can do to make our calculations a bit easier is to apply the initial condition to compute constants. At $t = 0$, we know $N = 100$. So, at $t = 0$

$$\frac{(100)}{500 - (100)} = C e^{500k(0)} = C e^0 = C$$

This means

$$C = 0.25$$

Also, we can use the information at $t = 4$ where $N = 200$. This means we want

$$\frac{(200)}{500 - (200)} = C e^{500k(4)} = (0.25) e^{2000k}$$

or

$$e^{2000k} = \frac{8}{3}$$

Then $k \approx 0.0005$. In the end the implicit relationship becomes

$$\frac{N}{500 - N} = (0.25)e^{(0.2452)t}$$

Solution Step 5:

The last step for this problem is to solve for N in the equation above. Clearing fractions by multiplying both sides by $500 - N$ implies

$$N = (0.25)e^{(0.2452)t}(500 - N) = 125e^{(0.2452)t} - (0.25)e^{(0.2452)t}N$$

or

$$N + (0.25)e^{(0.2452)t}N = N(1 + (0.25)e^{(0.2452)t}) = 125e^{(0.2452)t}$$

Finally, we can divide by the coefficient of N to obtain

$$N = \frac{125e^{(0.2452)t}}{1 + (0.25)e^{(0.2452)t}}$$