Problem Definition

Problem 25. Describe the intervals on which the following function is continuous.

$$f(x) = \begin{cases} 3 + x, & x \leq 2 \\ x^2 + 1 & x > 2 \end{cases}$$

Solution Step 1:

First, we should look at the function definition for $x \leq 2$ and $x > 2$. For values of $x \leq 2$ the function is defined to be the polynomial $x + 3$ which is continuous for all real numbers. So, the function defined above is continuous on the interval $(-\infty, 2)$. Also, the function is defined to be the polynomial $x^2 + 1$ for all $x > 2$. So our function is continuous on the interval $(2, \infty)$.

Solution Step 2:

The next step is to investigate what happens where the function definition changes; that is at $x = 2$. Since the function is defined at $x = 2$ by

$$f(2) = 2 + 3 = 5$$

The next step is to test to see if the limit exists. For this problem, we will need to evaluate the left and right hand limits. The left hand limit is given by

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x + 3) = (2 + 3) = 5$$

The right hand limit is computed as follows.

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x^2 + 1) = (2^2 + 1) = 5$$

Since the right and left hand limits exist and are equal to each other the limit exists and we can write

$$\lim_{x \to 2} f(x) = 5$$

The second property that needs to be satisfied is

$$\lim_{x \to 2} f(x) = f(2)$$

Since this is also true, the function is continuous at $x = 2$.

Solution Step 3:

Finally, we can combine our results to see that the function is continuous on the following.

$$(-\infty, 2) \bigcup \{2\} \bigcup (2, \infty) = (-\infty, \infty)$$

So, this function is continuous for all real numbers.