Problem Definition

Problem 45. Compute the equation of a line to the function

$$f(x) = -\frac{1}{4}x^2$$

that is parallel to the line defined by $x + y = 0$.

Solution Step 1:

In the first step, we will determine the slope of the line given at the end of the problem. Given that $x + y = 0$, we can easily solve for y in terms of x to see that

$$y = -x$$

The slope of this line is negative one.

Solution Step 2:

The next task is to find a point where the slope of the tangent line for the graph of the function given to us is negative one. The derivative can be computed using the definition of the derivative as follows.

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{-\frac{1}{4}(x+\Delta x)^2 + \frac{1}{4}x^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{-\frac{1}{4}((x+\Delta x)^2 + x^2)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left(-\frac{1}{4}\frac{((x+\Delta x)^2 + x^2)}{\Delta x}\right) \\
&= -\frac{1}{4}\lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - x^2}{\Delta x} \\
&= -\frac{1}{4}\lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2}{\Delta x} \\
&= -\frac{1}{4}\lim_{\Delta x \to 0} (2x + \Delta x) \\
&= -\frac{1}{2}x
\end{aligned}$$

So, the derivative we need is $f'(x) = -(1/2)x$.

Solution Step 3:

We want the derivative to be -1, so we solve the equation

$$f'(x) = -(1/2)x = -1$$

The only solution is $x = 2$ which can be seen after multiplying the last two terms by -2.

Solution Step 4:

For $x = 2$ we can compute the value $f(2) = (-1/4)(2^2) = -1$. This means the equation of the tangent line must have a slope of -1 and pass through the point $(2, -1)$. The point-slope from of a linear equation is of the form

$$y - y_0 = m(x - x_0)$$

In our case, we have $y_0 = -1$, $m = -1$, and $x_0 = 2$. Substituting the values into the point slope form of the equation gives

$$y + 1 = (-1)(x - 2)$$

or simplifying

$$y = -x + 1$$