Problem Definition

Problem 29. Find the points of inflection of the graph of the given function.

$$f(x) = (x-1)^3(x-5)$$

Solution Step 1:

The first step is to compute the derivative of the of the function.

$$\begin{aligned} f'(x) &= (3)(x-1)^2(x-5) + (x-1)^3(1) \\ &= (x-1)^2\left(3(x-5) + (x-1)\right) \\ &= (x-1)^2\left(4x-16\right) = 4(x-1)^2(x-4) \end{aligned}$$

Solution Step 2:

The function is a polynomial and the only critical points are the zeros of the derivative. For this problem, the zeros are $x = 1$ (two copies) and $x = 4$.

Solution Step 3:

So, let's try the second derivative test on the function. The second derivative is

$$\begin{aligned} f''(x) &= 8(x-1)(x-4) + 4(x-1)) \\ &= (x-1)(8(x-4) + 4(x-1)) \\ &= (x-1)(12x-36) = 12(x-1)(x-3) \end{aligned}$$

The product rule has been used and the result simplified.

Solution Step 4:

In this case, we need to find points where the second derivative changes sign. There are three intervals.

(a) The first interval is $(-\infty, 1)$. The second derivative at $x = 0$ is

$$f''(0) = 12(0-1)(0-3) = 36 > 0$$

So $f(x)$ is concave upward on this interval.

(b) The second interval is $(1,3)$. The second derivative at $x = 2$ is

$$f''(2) = 12(2-1)(2-3) = -12 < 0$$

So $f(x)$ is concave downward on this interval.

(c) The last interval is $(3,\infty)$. The second derivative at $x = 4$ is

$$f''(4) = 12(4-1)(4-3) = 36 > 0$$

So $f(x)$ is concave upward on this interval.

Since the concavity changes at both points, $x = 1$ and $x = 3$ are both points of inflection.

Solution Step 5:

To summarize, the points $(1,0)$ and $3,-36)$ on the graph of the function are inflection points.