Problem Definition

Problem 33. **Elasticity:** The demand function for a product is given by

$$x = p^2 - 20p + 100$$

(a) Consider a price of \$2. If the price increases 5%, what is the corresponding percentage in the quantity demanded?

(b) Average elasticity of demand is defined to be the percent change in quantity divided by the percent change in price. Use the percent in part (a) to find the average elasticity over the interval $[2, 2.1]$.

(c) Find the elasticity for a price of \$2 and compare the result to that in part (b).

(d) Find an expression for the total revenue and find the values of x and p that maximize the total revenue.

Solution Step 1:

For part (a) we need to compute the percentage change in x. When $p = 2$, $x = 4 - 40 + 100 = 64$ and when $p = 2.10$ (a 5% increase) $x = 62.41$. Then the percent increase is

$$\frac{x(2.1) - x(2.0)}{x(2.0)} = \frac{62.41 - 64}{64} \approx -0.0248$$

which translates to -2.48%

Solution Step 2:

For part (b) to divide the result from part by the percent change in price

$$\frac{\Delta x}{Deltap} \approx \frac{-0.0248}{(2.1 - 2.0)/2.0} \approx -0.0496$$

which translates to -4.96%

Solution Step 3:

For part (c) we need to compare the price elasticity to the result in part (b). The price elasticity for $p = \$2$ is computed as follows. The price elasticity is given by

$$\eta = \frac{p/x}{dp/dx}$$

This means we need dp/dx. We can implicitly differentiate the price demand relationship as follows.

$$1 = 2p\frac{dp}{dx} - 20\frac{dp}{dx}(2p-20)\frac{dp}{dx}$$

without solving for p in terms of x. Solving for the derivative we get

$$\frac{dp}{dx} = \frac{1}{2p-20}$$

So, the formula for price elasticity for this problem is

$$\eta = \frac{p/(p^2-20p+100)}{1/(2p-20)} = \frac{p(2p-20)}{p^2-20p+100}$$

If we evalaute this formula for a price of \$2 we find

$$\eta = \frac{(2)(2(2)-20)}{(2)^2-20(2)+100} = -0.05$$

or $\eta = -5\% \approx -4.96\%$.

Solution Step 4:

For part (d) we need the revenue as a function of either price, p, or demand, x. That is,

$$R = xp = (p^2 - 20p + 100)p = p^3 - 20p^2 + 100p$$

This is easier than using the demand variable. Differentiating this with respect to p gives

$$R' = 3p^2 - 40p + 100$$

The roots of this polynomial can be obtained with the quadratic formula and are

$$p = \frac{40 \pm \sqrt{40^2 - 4(3)(100)}}{2(3)} = \frac{40 \pm \sqrt{400}}{6} = \frac{20}{3} \pm \frac{10}{3}$$

That is, $p = \$3.33$ or $p = \$10$. For these cases, we have

$$x = (10/3)^2 - 20(10/3) + 100 = 400/9$$

and

$$x = (10)^2 - 20(10) + 100 = 0$$

respectively. Certainly, the second case with $p = \$10$ is not the maximizer, so we use $p = \$3.33$.

Solution Step 5:

In summary, we have

price elasticity	$\eta = -0.5$
price that maximizing revenue	$p = -3.33$
demand that maximizing revenue	$x = 400/9$
maximum revenue	$R = \$4000/27$