Problem Definition

Problem 23. Find the indefinite integral and check your result by differentiation.

$$\int 5u\sqrt[3]{1-u^2}\ du$$

Solution Step 1:

The use of a substitution can transform the integral into an easier integral. If we choose to do the work, the substitution should address a complicated term. For that reason, let's go after the cube root in the expression. So, let's try

$$v = 1 - u^2$$

This means that $dv = -2udu$. Using these expressions we get

$$\begin{aligned}\int 5u\sqrt[3]{1-u^2}\ du &= \int \frac{5}{(-2)}(-2)u\sqrt[3]{1-u^2}\ du \\ &= -\frac{5}{2}\int \sqrt[3]{1-u^2}\ (-2u\ du) \\ &= -\frac{5}{2}\sqrt[3]{v}\ dv\end{aligned}$$

Solution Step 2:

Using the transformed integral, we can compute the indefinite integral in terms of the variable v.

$$\begin{aligned}\int -\frac{5}{2}\sqrt[3]{v}\ dv &= -\frac{5}{2}\int v^{1/3}\ dv \\ &= -\frac{5}{2}\left(\frac{1}{4/3}\ v^{4/3}\right) + C \\ &= -\frac{15}{8}v^{4/3} + C\end{aligned}$$

This gives the antiderivative in terms of v.

Solution Step 3:

The last step is to transform the answer from the variable v back to the original variable u.

$$\int 5u\sqrt[3]{1-u^2}\ du = -\frac{15}{8}(1-u^2)^{4/3} + C$$

Solution Step 4:

Finally, the problem asks us to verify the solution by differentiating the antiderivative. The calculations are the following.

$$\begin{aligned}
\frac{d}{dx}\left(-\frac{15}{8}(1-u^2)^{4/3} + C\right) &= \frac{d}{dx}\left(-\frac{15}{8}(1-u^2)^{4/3}\right) + \frac{d}{dx}(C) \\
&= -\frac{15}{8}\frac{d}{dx}\left((1-u^2)^{4/3}\right) + (0) \\
&= -\frac{15}{8}\frac{4}{3}\left((1-u^2)^{1/3}\right)(-2u) \\
&= 5\left((1-u^2)^{1/3}\right)u \\
&= 5u(1-u^2)^{1/3}
\end{aligned}$$

This is the integrand in the original problem.