Problem Definition

Problem 43. **Consumer and Producer Surplus** Find the consumer and producer surplus for the case below.

$$p_1(x) = 200 - 0.02x^2 \qquad p_2(x) = 100 + x$$

Solution Step 1:

The starting point in determining a solution is to find the intersection of the two price functions. By setting $p_1(x) = p_2(x)$ we obtain the following equation x.

$$200 - 0.02x^2 = 100 + x$$

or multiplying both sides by 50 the equation becomes

$$10000 - x^2 = 5000 + 50x$$

Solving for x gives

$$0 = -10000 + x^2 + 5000 + 50x = x^2 + 50x - 5000 = (x + 100)(x - 50)$$

The two roots are $x = -100$ and $x = 50$. Since we expect to produce positive numbers of items and it is impossible to produce a negative demand, the only root that makes sense is the root $x = 50$. This is the equilibrium demand and the associated price is the equilibrium price. That is, $p_2(50) = 100 + 50 = \$150$.

Solution Step 2:

The next step is to determine the price function that is above the equilibrium price of \$150. The price function $p_1(x) \geq 150$ for all $x \leq 50$. Also, the price function $p_2(x) = 100 + x \leq 150$ for all $x \leq 0$. The consumer surplus lies above the equilibrium and is computed using the definite integral

$$\begin{aligned}
\text{Consumer Surplus} &= \int_0^{50} (p_1(x) - 150)\, dx \\
&= \int_0^{50} \left(200 - 0.02x^2 - 150\right) dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{50} \left(50 - 0.02x^2\right) dx \\
&= \left(50x - \frac{0.02}{3}\ x^3\right) |_0^{50} \\
&= \left(50(50) - \frac{0.02}{3}\ (50)^3\right) - \left(50(0) - \frac{0.02}{3}\ (0)^3\right) \\
&= \left(2500 - \frac{2500}{3}\ \right) - (0) \\
&= \frac{5000}{3} \approx 1666.67
\end{aligned}
$$

The producer surplus is given by

$$
\begin{aligned}
\text{Producer Surplus} &= \int_0^{50} (150 - p_2(x))\, dx \\
&= \int_0^{50} (150 - (100 + x))\, dx \\
&= \int_0^{50} (50 - x)\, dx \\
&= \left(50x - \frac{x^2}{2}\right) |_0^{50} \\
&= \left(50(50) - \frac{(50)^2}{2}\right) - \left(50(0) - \frac{(0)^2}{2}\right) \\
&= \left(2500 - \frac{2500}{2}\right) - (0) \\
&= 2500 - 1250 = 1250
\end{aligned}
$$