Problem Definition

Problem 23. Find the indefinite integral. (Hint: Integration by parts is not necessary in all cases in this section of the text.

$$\int x\sqrt{x-1}\;dx$$

Solution Step 1:

In this case, it is not necessary to compute the indefinite integral using integration by parts. Substituting for the argument in the square root will give an easier integral with which to work. We can use

$$y = x - 1$$

and require $dy = dx$ and $x = y + 1$. The new form of the integral is

$$\int (y+1)\sqrt{y}\;dy = \int \left(y^{3/2} + y^{1/2}\right)\,dy$$

Solution Step 2:

The indefinite integral is then computed as follows.

$$\begin{aligned}\int \left(y^{3/2} + y^{1/2}\right)\,dy &= \left(\frac{y^{5/2}}{5/2} + \frac{y^{3/2}}{3/2}\right) + C \\ &= \frac{2}{5}\,y^{5/2} + \frac{2}{3}\,y^{3/2} + C\end{aligned}$$

Solution Step 3:

The last step is to return to the original variables.

$$\frac{2}{5}\,y^{5/2} + \frac{2}{3}\,y^{3/2} + C = \frac{2}{5}\,(x-1)^{5/2} + \frac{2}{3}\,(x-1)^{3/2} + C$$

Note that integration by parts will work, but is a bit more difficult.

Solution Step 4:

To check the work, we can differentiate the solution in an effort to reproduce the integrand.

$$\begin{aligned}
\frac{d}{dx}\left(\frac{2}{5}\,(x-1)^{5/2}\right. &+ \left.\frac{2}{3}\,(x-1)^{3/2}+C\right) \\
&= \frac{d}{dx}\left(\frac{2}{5}\,(x-1)^{5/2}\right)+\frac{d}{dx}\left(\frac{2}{3}\,(x-1)^{3/2}\right)+\frac{d}{dx}\,(C) \\
&= \frac{2}{5}\frac{d}{dx}\left((x-1)^{5/2}\right)+\frac{2}{3}\,\frac{d}{dx}\left((x-1)^{3/2}\right) \\
&= \left(\frac{2}{5}\right)\left(\frac{5}{2}\right)\left((x-1)^{3/2}\right)+\left(\frac{2}{3}\right)\left(\frac{3}{2}\right)\left((x-1)^{1/2}\right) \\
&= (x-1)^{3/2}+(x-1)^{1/2} \\
&= (x-1)^{1/2}\left((x-1)+1\right) \\
&= x(x-1)^{1/2} \\
&= x\sqrt{x-1}
\end{aligned}$$

This is the same function we started with.