Problem Definition

Problem 37. Evaluate the definite integral.

$$\int_0^1 \frac{x^3}{x^2 - 2} dx$$

Solution Step 1:

The first step in the process is to do a little algebra on the integrand. The first step is to check the degree of the polynomial in the numerator against the degree of the polynomial in the denominator. Since the degree of the polynomial in the denominator (two) is less than the degree of the polynomial in the numerator (three) we need to perform a long division before doing the partial fraction decomposition. The division will produce

$$\frac{x^3}{x^2 - 2} = x + \frac{2x}{x^2 - 2}$$

Solution Step 2:

The remainder from the previous step needs to be broken down a bit further. The basic form is

$$\frac{2x}{x^2 - 2} = \frac{2x}{(x + \sqrt{2})(x - \sqrt{2})} = \frac{A}{x + \sqrt{2}} + \frac{B}{x - \sqrt{2}}$$

Clearing fractions gives the following equation

$$2x = A(x - \sqrt{2}) + B(x + \sqrt{2})$$

One way to proceed is to set x to each of the roots and solve for the coefficients A and B. Setting $x = \sqrt{2}$ results in

$$2\sqrt{2} = A(\sqrt{2} - \sqrt{2}) + B(\sqrt{2} + \sqrt{2})$$

or

$$2\sqrt{2} = 2B\sqrt{2} \qquad \rightarrow \qquad B = 1$$

Then setting $x = -\sqrt{2}$ results in the equation

$$-2\sqrt{2} = A(-\sqrt{2} - \sqrt{2}) + B(-\sqrt{2} + \sqrt{2})$$

or

$$-2\sqrt{2} = -2A\sqrt{2} \qquad \rightarrow \qquad A = 1$$

So, the remainder from the first step can be written as

$$\frac{2x}{x^2-2} = \frac{1}{x+\sqrt{2}} + \frac{1}{x-\sqrt{2}}$$

Solution Step 3:

The last step is to compute the integral.

$$\begin{aligned}
\int_0^1 \frac{x^3}{x^2-2}dx &= \int_0^1 \left(x + \frac{1}{x+\sqrt{2}} + \frac{1}{x-\sqrt{2}}\right) dx \\
&= \left(\frac{x^2}{2} + ln|x+\sqrt{2}| + ln|x-\sqrt{2}|\right) |_0^1 \\
&= \left(\frac{x^2}{2} + ln|(x+\sqrt{2})(x-\sqrt{2})|\right) |_0^1 \\
&= \left(\frac{x^2}{2} + ln|(x^2-2)|\right) |_0^1 \\
&= \left(\frac{1}{2} + ln|((1)^2-2)|\right) - \left((0) + ln|((0)^2-2)|\right) \\
&\approx -0.193
\end{aligned}$$

We should be careful to note that this is a place where the rule

$$\int \frac{1}{x}dx = ln|x| + C$$

with the absolute values on the argument of the logarithm are absolutely necessary.