Problem Definition

Problem 7. Determine whether or not the improper integral converges. If it does, evaluate the integral.

$$\int_5^\infty \frac{x}{\sqrt{x^2-16}}\, dx$$

Solution Step 1:

The integral is an improper integral since one of the limits of integration is ∞. We can write the definite integral as a limit as follows.

$$\int_5^\infty \frac{x}{\sqrt{x^2-16}}\, dx = \lim_{b\to\infty} \int_5^b \frac{x}{\sqrt{x^2-16}}\, dx$$

If the limit exists, then the improper integral exists and we can try to compute the value.

Solution Step 2:

Let's start by computing an antiderivative of

$$\int \frac{x}{\sqrt{x^2-16}}\, dx$$

using the substitution

$$u = x^2 - 16$$

which requires $dy = 2xdx$. These definitions can be used to transform the integral as follows.

$$\begin{aligned}
\int \frac{x}{\sqrt{x^2-16}}\, dx &= \frac{1}{2}\int \frac{2x}{\sqrt{x^2-16}}\, dx \\
&= \frac{1}{2}\int \frac{1}{\sqrt{x^2-16}}\, (2xdx) \\
&= \frac{1}{2}\int \frac{1}{\sqrt{u}}\, du \\
&= \frac{1}{2}\int u^{-1/2}\, du \\
&= \frac{1}{2}\frac{u^{1/2}}{1/2} + C \\
&= u^{1/2} + C
\end{aligned}$$

Transforming back to the original variable x gives us

$$\int \frac{x}{\sqrt{x^2 - 16}}\, dx = (x^2 - 16)^{1/2} + C$$