

Problem Definition

Problem 59. Find the second partial derivatives of

$$z = \frac{xy}{x - y}$$

That is, compute

$$\frac{\partial^2 z}{\partial x^2}, \quad \frac{\partial^2 z}{\partial x \partial y}, \quad \frac{\partial^2 z}{\partial y \partial x}, \quad \frac{\partial^2 z}{\partial y^2}$$

Solution Step 1:

The first step is to compute the first partial derivatives. These derivatives can be computed using the quotient rule applied to each of the variables.

$$\begin{aligned} z_x &= \frac{\partial}{\partial x} \left[\frac{xy}{x - y} \right] \\ &= \frac{y(x - y) - (xy)(1)}{(x - y)^2} \\ &= \frac{yx - y^2 - xy}{(x - y)^2} \\ &= \frac{-y^2}{(x - y)^2} \end{aligned}$$

The other derivative is

$$\begin{aligned} z_y &= \frac{\partial}{\partial y} \left[\frac{xy}{x - y} \right] \\ &= \frac{x(x - y) - (xy)(-1)}{(x - y)^2} \\ &= \frac{x^2 - xy + xy}{(x - y)^2} \\ &= \frac{x^2}{(x - y)^2} \end{aligned}$$

Solution Step 2:

We can now compute the second partial derivatives using the following calculations.

$$\begin{aligned}
 z_{xx} &= \frac{\partial}{\partial x} \left[\frac{-y^2}{(x-y)^2} \right] \\
 &= \frac{(0)(x-y)^2 + y^2(2)(x-y)(1)}{(x-y)^4} \\
 &= \frac{2y^2(x-y)}{(x-y)^4} \\
 &= \frac{2y^2}{(x-y)^3} \\
 \\
 z_{xy} &= \frac{\partial}{\partial y} \left[\frac{-y^2}{(x-y)^2} \right] \\
 &= \frac{(-2y)(x-y)^2 - y^2(2)(x-y)(1)}{(x-y)^4} \\
 &= \frac{(-2y)(x-y) - 2y^2}{(x-y)^3} \\
 &= \frac{-2yx + 2y^2 - 2y^2}{(x-y)^3} \\
 &= -\frac{2yx}{(x-y)^3}
 \end{aligned}$$

Notice that the last partial derivative and the next derivative computed are identical. This will be true for all functions that are smooth enough. If the derivatives are not continuous, the order of differentiation may make a difference.

$$\begin{aligned}
 z_{yx} &= \frac{\partial}{\partial x} \left[\frac{x^2}{(x-y)^2} \right] \\
 &= \frac{(2x)(x-y)^2 - x^2(2)(x-y)(1)}{(x-y)^4} \\
 &= \frac{(2x)(x-y) - 2x^2}{(x-y)^3} \\
 &= \frac{2x^2 - 2xy - 2x^2}{(x-y)^3}
 \end{aligned}$$

$$= -\frac{2yx}{(x-y)^3}$$

$$\begin{aligned} z_{yy} &= \frac{\partial}{\partial y} \left[\frac{x^2}{(x-y)^2} \right] \\ &= \frac{(0)(x-y)^2 - x^2(2)(x-y)(-1)}{(x-y)^4} \\ &= \frac{2x^2}{(x-y)^3} \end{aligned}$$