Problem Definition

Problem 13. Examine each function for relative extrema and saddle points.

$$f(x, y) = x^2 - y^2 + 4x - 4y - 8$$

Solution Step 1:

The first step in this problem is to find critical points for the function. These occur where the first partial derivatives are simultaneously zero. The first partial derivatives are

$$f_x(x, y) = \frac{\partial}{\partial x}\left[x^2 - y^2 + 4x - 4y - 8\right] = 2x + 4$$

and

$$f_y(x, y) = \frac{\partial}{\partial y}\left[x^2 - y^2 + 4x - 4y - 8\right] = -2y - 4$$

Solution Step 2:

Setting both partial derivatives to zero results in the following linear system of equations.

$$\begin{aligned} 2x + 4 &= 0 \\ -2y - 4 &= 0 \end{aligned}$$

The only solution for this pair of equations will be $x = -2$ and $y = -2$. So the only critical point for the function is $(2, 2)$.

Solution Step 3:

To use the second derivative test we need the second partial derivatives. These are

$$f_{xx}(x, y) = 2, \quad f_{xy}(x, y) = f_{yx}(x, y) = 0, \quad \text{and } f_{yy}(x, y) = -2$$

Solution Step 4:

The analysis will require the evaluation of the discriminant

$$d = f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2$$

at each critical point. The discriminant for an arbitrary point for this function is

$$d = (2)(-2) - (0)^2 = -4$$

The value of d is a constant less than zero. Therefore, the critical point is a saddle point for the function.

Solution Step 5:

To summarize, there is only one critical point at $(x, y) = (-2, -2)$. This is a saddle point. Since there are no other critical points, the function has no relative extrema.