Problem Definition

Problem 27. Find the critical points and test for relative extrema. List the critical points for which the second partial derivative test fails.

$$f(x,y) = x^3 + y^3$$

Solution Step 1:

The first step in this problem is to find critical points for the function. These occur where the first partial derivatives are simultaneously zero. The first partial derivatives are

$$f_x(x,y) = \frac{\partial}{\partial x}\left[x^3 + y^3\right] = 3x^2$$

and

$$f_y(x,y) = \frac{\partial}{\partial y}\left[x^3 + y^3\right] = 3y^2$$

Solution Step 2:

Setting both partial derivatives to zero results in the following linear system of equations.

$$\begin{aligned} 3x^2 &= 0 \\ 3y^2 &= 0 \end{aligned}$$

The only solution for this pair of equations will be $x = 0$ and $y = 0$. So the only critical point for the function is $(0,0)$.

Solution Step 3:

To use the second derivative test we need the second partial derivatives. These are

$$f_{xx}(x,y) = 6x, \quad f_{xy}(x,y) = f_{yx}(x,y) = 0, \quad \text{and } f_{yy}(x,y) = 6y$$

Solution Step 4:

The analysis will require the evaluation of the discriminant

$$d = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2$$

at each critical point. The discriminant for an arbitrary point for this function is

$$d = (6x)(6y) - (0)^2 = 36xy$$

The value of d is zero at the only critical point $(x, y) = (0, 0)$. This means that the second derivative test does not provide any information. It turns out that $(0, 0)$ is again a saddle point.

Solution Step 5:

Let's show that this is truely a saddle point. Suppose we set $y = 0$ and look at points along the x axis. If $x < 0$, $f_x(0, 0) = 3x^2 > 0$. The same is true for all $x > 0$. So, the critical point acts like an inflection point along the x axis.