Problem Definition

Problem 37. **Revenue** A company manufactures two products. The total revenue from x_1 units of product 1 and x_2 units of product 2 is given by

$$R = -5x_1^2 - 8x_2^2 - 2x_1x_2 + 42x_1 + 102x_2$$

Solution Step 1:

The first step in this problem is to find critical points for the function. These occur where the first partial derivatives are simultaneously zero. The first partial derivatives are

$$R_{x_1}(x_1, x_2) = -10x_1 - 2x_2 + 42$$

and

$$R_{x_2}(x_1, x_2) = -16x_2 - 2x_1 + 102$$

Solution Step 2:

Setting both partial derivatives to zero results in the following linear system of equations.

$$\begin{aligned} -10x_1 - 2x_2 + 42 &= 0 \\ -16x_2 - 2x_1 + 102 &= 0 \end{aligned}$$

or

$$\begin{aligned} 10x_1 + 2x_2 &= 42 \\ 16x_2 + 2x_1 &= 102 \end{aligned}$$

Dividing both equations by 2 gives

$$\begin{aligned} 5x_1 + x_2 &= 21 \\ 8x_2 + x_1 &= 51 \end{aligned}$$

Solving the first equation for x_2 gives $x_2 = 21 - 5x_1$. Substituting this into the second equation results in

$$8(21 - 5x_1) + x_1 = 51$$

or

$$117 - 39x_1 = 0$$

This gives $x_1 = 3$ and using the solution for x_2 gives $x_2 = 21 - 5(3) = 6$. Since there is only one solution for the linear system we find a single critial point $(3, 6)$.

Solution Step 3:

To use the second derivative test we need the second partial derivatives. These are

$$R_{x_1,x_1}(x_1, x_2) = -10, \quad R_{x_1,x_2}(x_1, x_2) = -2 \text{ and } R_{x_2,x_2}(x_1, x_2) = -16$$

Solution Step 4:

The analysis will require the evaluation of the discriminant

$$d = R_{x_1,x_1}(x_1, x_2)R_{x_2,x_2}(x_1, x_2) - (R_{x_1,x_2}(x_1, x_2))^2$$

at each critical point. The discriminant for an arbitrary point for this function is

$$d = (-10)(-16) - (-2)^2 = 154$$

The value of d is greater zero at the only critical point $(x_1, x_2) = (3, 6)$. This means that the second derivative test indicates that the critical point is a local maximum or minimum. Since $R_{x_1,x_1} < 0$ the critical point is a local maximum. The negative derivative is locally concave downward.

Solution Step 5:

The end result is the revenue is maximized for $x_1 = 3$ and $x_2 = 6$. The revenue at this point is

$$R = -5(3)^2 - 8(6)^2 - 2(3)(6) + 42(3) + 102(6) = 309$$