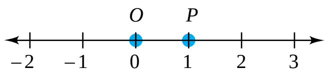
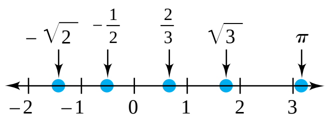
1.3 REAL NUMBER PROPERTIES; COMPLEX NUMBERS

> . . . one of the central themes of science [is] the mysterious power of mathematics to prepare the ground for physical discoveries which could not have been foreseen by the mathematicians who gave the concepts birth.
>
> *Freeman Dyson*



(a) Integers on a number line.



(b) Real numbers on a number line.

FIGURE 5

Real Number Line

One of the great ideas in the history of mathematics is that the set of real numbers can be associated with the set of points on a line. We assume a one-to-one correspondence that associates each real number with exactly one point on a line, and every point on the line corresponds to exactly one real number.

We frequently identify a number with its point, and vice versa, speaking of "the point 2" rather than "the point that corresponds to 2." Figure 5 shows a few numbers and the corresponding points on a number line.

Order Relations and Intervals

The number line also represents the ordering of the real numbers. We assume that the ideas of less than and greater than, and the following notation are familiar:

Order relations for real numbers

Notation	*Terminology*	*Meaning*
$b < c$	b is less than c.	$c = b + d$, for some positive number d
$c > b$	c is greater than b	$b < c$
$b \le c$	b is less than or equal to c.	$b < c$ or $b = c$
$c \ge b$	c is greater than or equal to b.	$b \le c$

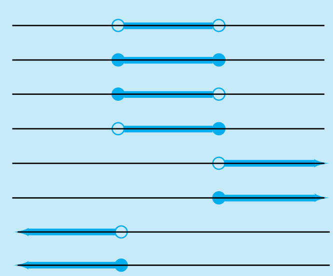
> ***T****he mathematics course at San Diego High School was standard for that time: plane geometry in the tenth grade, advanced algebra in the eleventh, and trigonometry and solid geometry in the twelfth. . . . After plane geometry, I was the only girl still taking mathematics.*
>
> Julia Robinson

We also need notation for sets of all numbers between two given numbers, or all numbers less than or greater than a given number. Such sets are called **intervals.**

Definition: intervals

Suppose b and c are real numbers and $b < c$:

Name	*Notation*	*Number-line diagram*
Open interval	$(b, c) = \{x \mid b < x < c\}$	
Closed interval	$[b, c] = \{x \mid b \le x \le c\}$	
Half-open interval	$[b, c) = \{x \mid b \le x < c\}$	
Half-open interval	$(b, c] = \{x \mid b < x \le c\}$	
Infinite intervals	$(b, \infty) = \{x \mid x > b\}$	
	$[b, \infty) = \{x \mid x \ge b\}$	
	$(-\infty, b) = \{x \mid x < b\}$	
	$(-\infty, b] = \{x \mid x \le b\}$	



In these definitions, the symbol ∞ (infinity) *does not represent a number,* and we never use a closed bracket to indicate that ∞ is included in an infinite interval.

Absolute Value and Distance

We have no difficulty in finding the absolute value of specific numbers, as in

$$|2| = 2, \qquad |0| = 0, \qquad |-1.375| = 1.375.$$

There are always *two numbers* having the same nonzero absolute value, as

$$|2| = |-2| = 2, \qquad \text{and} \qquad \left|\frac{-\pi}{3}\right| = \left|\frac{\pi}{3}\right| = \frac{\pi}{3}.$$

Following the pattern of the above examples, *the absolute value of a positive number is itself; the absolute value of a negative number is its opposite.*

In working with an expression like $|2x - 3|$, the quantity $2x - 3$ is neither positive nor negative until we give a value to x. All we can say is that $|2x - 3|$ is either $2x - 3$ or its opposite, $-(2x - 3)$. Thus

$|2x - 3| = 2x - 3$ for all the x-values that make $2x - 3$ positive $(2, \frac{9}{5}, 5\pi, \ldots)$,

$|2x - 3| = -2x + 3$ for all the x-values that make $2x - 3$ negative $(1, \frac{\pi}{3}, 0, \ldots)$.

Looking at a number line, the numbers satisfying $|x| = 3$ are 3 and -3, the two numbers *whose distance from 0 is 3*. More generally, the numbers located two units from 7 are the numbers 5 and 9, and $|7 - 5| = 2$, and $|7 - 9| = 2$.

These examples lead to two ways of looking at absolute values, both of which are useful, so we include them in our definition.

Definition: absolute value

For any expression u, the ***absolute value of u,*** denoted by $|u|$, is given by

$$|u| = \begin{cases} u & \text{if} \quad u \geq 0 \\ -u & \text{if} \quad u < 0 \end{cases}$$

If a and b are any real numbers, then ***the distance between a and b*** is given by

$$|a - b| = |b - a|.$$

It follows that $|a| = |a - 0|$ is ***the distance between a and 0.***

Calculators and Absolute Value

In finding the absolute value of any particular number, we shouldn't have to rely on a calculator, but a calculator can be helpful nonetheless.

We know, for example, that $|\pi - \sqrt{10}|$ is either $\pi - \sqrt{10}$ or $\sqrt{10} - \pi$, depending on which is positive. We do not need to use the calculator function ABS; in fact, if we were to try (see the Technology Tip in Section 1.5 for suggestions about how to enter ABS), we would find only that

$$\text{ABS}(\pi - \sqrt{10}) \approx 0.020685.$$

This is true, but not helpful in deciding whether $|\pi - \sqrt{10}|$ is equal to $\pi - \sqrt{10}$ or $\sqrt{10} - \pi$. If, however, we use the calculator to learn that $\pi - \sqrt{10}$ is negative, about -0.020685, then we immediately know that $|\pi - \sqrt{10}| = \sqrt{10} - \pi$.

▶EXAMPLE 1 *Absolute value*

(a) If $t = 1 - \sqrt{3}$, show both t and $-t$ on a number line and express $|t|$ in exact form without using absolute values. **(b)** Find all numbers x such that $|2x - 3| = 1$.

Solution

(a) Since t is negative ($t \approx -.0732$), $|t|$ is the opposite of t:

$$|t| = |1 - \sqrt{3}| = -(1 - \sqrt{3}) = \sqrt{3} - 1.$$

Both t and $-t$ are shown on the number line in Figure 6.

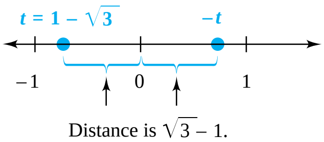


FIGURE 6

(b) The two numbers whose absolute value is 1 are 1 and -1. Thus, if $|2x - 3| = 1$, we have either

$$2x - 3 = 1 \qquad \text{or} \qquad 2x - 3 = -1.$$

Solving each, we have

$$x = 2 \qquad \text{or} \qquad x = 1.$$ ◀

Some useful properties of absolute values

Suppose x and y are any real numbers.

1. $|x| \geq 0$	**2.** $|x| = |-x|$
3. $|x \cdot y| = |x| \cdot |y|$	**4.** $\sqrt{x^2} = |x|$
5. $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ if $y \neq 0$	**6.** $|x + y| \leq |x| + |y|$

▶EXAMPLE 2 *Absolute value arithmetic*

Let $x = -3$, $y = 2$, and $z = 1 - \sqrt{3}$. Evaluate the expressions. Are the values in each pair equal or not?

(a) $\sqrt{x^2}, x$ **(b)** $|x \cdot y|, |x| \cdot |y|$ **(c)** $|y + z|, |y| + |z|$
(d) $|x + z|, |x| + |z|$

Strategy: Identify each number as positive or negative before applying a definition of absolute value.

Solution

(a) $\sqrt{x^2} = \sqrt{(-3)^2} = \sqrt{9} = 3$; $x = -3$.
(b) $|x \cdot y| = |(-3) \cdot 2| = |-6| = 6$; $|x| \cdot |y| = |-3| \cdot |2| = 3 \cdot 2 = 6$.
(c) $|y + z| = |2 + (1 - \sqrt{3})| = |3 - \sqrt{3}| = 3 - \sqrt{3} \approx 1.27$.
$|y| + |z| = |2| + |1 - \sqrt{3}| = 2 + (\sqrt{3} - 1) = 1 + \sqrt{3} \approx 2.73$.
(d) $|x + z| = |-3 + (1 - \sqrt{3})| = |-2 - \sqrt{3}| = 2 + \sqrt{3} \approx 3.73$.
$|x| + |z| = |-3| + |1 - \sqrt{3}| = 3 + (\sqrt{3} - 1) = 2 + \sqrt{3}$.

The pairs in parts (b) and (d) are equal; those in (a) and (c) are not. ◀

Complex Numbers

Although most of our work deals exclusively with real numbers, sometimes we must expand to a larger set, the set of **complex numbers.** We need complex numbers mostly in two settings: for the solutions of polynomial equations, and for some trigonometric applications (Chapter 7). For the time being, all the information we need is simple complex-number arithmetic and how to take square roots. We include a picture of the complex plane and some properties of complex numbers for reference.

HISTORICAL NOTE GROWTH OF THE NUMBER SYSTEM

Boethius (left), using written Arabic numerals, triumphs over Pythagoras and his abacus in a mathematical contest. The goddess Arithmetica presides over the competition.

The ancient Greeks believed numbers expressed the essence of the whole world. Numbers to the Pythagorean philosophers meant whole numbers and their ratios—what we would call the *positive rational numbers.* It was extremely distressing to some when they discovered that something as simple as the diagonal of a square cannot be expressed rationally in terms of the length of the side of the square. Pythagoras (ca. 550 B.C.) is said to have sacrificed 100 oxen in honor of the discovery of irrational numbers. Nonetheless, irrational numbers were called *alogos* in Greek, carrying the double meaning that such numbers were not ratios and also that they were not to be spoken.

Hundreds of years passed before mathematicians became comfortable with the use of numbers like $\sqrt{2}$. Even we are reluctant to accept new numbers, as our language reflects. We equate rational with reasonable, and dislike irrational or negative concepts.

Not until the Middle Ages did mathematicians become secure with fractions and negative numbers. Also at that time, they recognized that irrationals have negatives, so two numbers have the square 2, namely $\sqrt{2}$ and $-\sqrt{2}$.

Complex numbers have a history somewhat shorter than that of irrational numbers. Cardan made the first public use of complex numbers in 1545 when he showed how to find two numbers with a sum of 10 and a product of 40, giving the result as $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$. Although he observed that the product equals $5^2 - (-15)$ or 40, he considered such expressions no more real than negative numbers describing lengths of line segments. In 1777 Euler first used the symbol i to denote $\sqrt{-1}$. As with $\sqrt{2}$, people first considered only one root of -1, but then recognized that i and $-i$ both satisfy the equation $x^2 + 1 = 0$.

We have come gradually to recognize that our number system is much larger and richer than childhood experience conceives. We extend our counting numbers to accommodate subtraction and division and then to solve simple equations such as $x^2 = 2$ and $x^2 = -1$. Other extensions are possible, and we hope to be open to accepting whatever is useful for solving new problems.

We are familiar with the fact that whenever we take the square of a nonzero real number, we always get a *positive* number. There is no real number that satisfies the simple equation $x^2 + 1 = 0$. Accordingly, we extend the real number system by introducing a new number i, whose distinguishing characteristic is that its square equals -1, $i^2 = -1$, which is sufficient to define the set C of complex numbers. See the Historical Note, "Growth of the Number System."

Definition: the set of complex numbers

$$C = \{c + di \mid c \text{ and } d \text{ are real numbers, and } i^2 = -1\}.$$

The set of complex numbers is really an extension of the set of real numbers because for any real number a, $a = a + 0i$. This observation gives the conclusion:

Every real number is also a complex number.

The Quadratic Formula, Complex Numbers, and Principal Square Roots

The roots of a quadratic equation may or may not be real numbers. For solving such an equation, we rely on another familiar tool from introductory algebra, the **quadratic formula.**

Quadratic formula

The roots of the equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression $b^2 - 4ac$ is called the **discriminant** of the equation and determines the nature of the roots.

If $b^2 - 4ac = 0$, there is *only one root,* given by $x = \dfrac{-b}{2a}$.
If $b^2 - 4ac > 0$, the quadratic formula gives *two real roots.*
If $b^2 - 4ac < 0$, there are *two nonreal complex roots.*

To apply the quadratic formula in the case with a negative discriminant, we need to extend the idea of square roots from our definition in Section 1.2 to the following.

Definition: principal square root

Suppose p is a positive real number. Then the **principal square roots** of p and $-p$ are given by:

$\sqrt{p}$ is the *nonnegative number whose square is p,* as $\sqrt{4} = 2$.
$\sqrt{-p}$ is the *complex number* $\sqrt{p}i$, as $\sqrt{-9} = 3i$.

For example, if $x^2 - 4x + 5 = 0$, the discriminant is negative and the quadratic formula gives the roots as $x = \dfrac{4 \pm \sqrt{-4}}{2} = \dfrac{4 \pm 2i}{2} = 2 \pm i$. The nonreal complex roots given by the quadratic formula always occur in what are called **conjugate pairs,** in this case $2 + i$ and $2 - i$. In general, for the complex number $z = c + di$, the **conjugate** of z, denoted by $\bar{z}$, is given by $\bar{z} = c - di$.

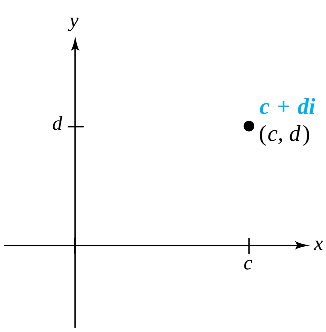


FIGURE 7
The complex number $c + di$ is identified with the point (c, d).

The Complex Plane

Just as we identify each real number with a point on a number line, we identify each complex number $c + di$ with a point in the plane having coordinates (c, d). See Figure 7. In this correspondence, the x-axis is the real number line and all real multiples of i are located on the y-axis.

The **standard form** for a complex number is $c + di$, where c and d are real numbers. The number c is called the **real part** and d is the **imaginary part.** If the imaginary part is nonzero, then we call the complex number $c + di$ a **nonreal-complex number.**

Complex Number Arithmetic

When are two complex numbers equal? How do we add, subtract, multiply, and divide complex numbers? Given complex numbers z and w in standard form, say $z = a + bi$ and $w = c + di$, we treat these numbers as we would any algebraic expressions, combining like terms in the usual fashion, with one exception. In multiplication we replace i^2 by -1 wherever it occurs. For division, $\frac{z}{w}$ $(w \neq 0)$, we multiply numerator and denominator by $\overline{w}$ as follows:

$$\frac{z}{w} = \frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di}$$

$$= \frac{ac - adi + bci - bdi^2}{c^2 - d^2 i^2} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}$$

Definitions: complex number arithmetic

Suppose $z = a + bi$, $w = c + di$, where a, b, c, and d are real numbers.

Equality: $z = w$ if and only if $a = c$ and $b = d$.
Addition: $z + w = (a + c) + (b + d)i$
Subtraction: $z - w = (a - c) + (b - d)i$
Multiplication: $z \cdot w = (ac - bd) + (ad + bc)i$
Division: $\frac{z}{w} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} i,$
where $c^2 + d^2 \neq 0$.

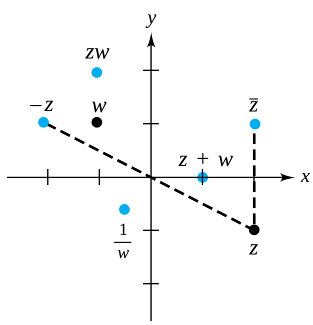


FIGURE 8
Points in the complex plane.

►EXAMPLE 3 Complex number arithmetic If $z = 2 - i$ and $w = -1 + i$, write each expression as a complex number in standard form and locate each on a diagram of the complex plane:

(a) $z + w$ **(b)** $\overline{z}$ and $-z$ **(c)** zw **(d)** $\frac{1}{w}$

Solution

(a) $z + w = (2 - i) + (-1 + i) = 1 + 0i = 1.$
(b) $\overline{z} = 2 + i$, and $-z = -2 + i$.
(c) $zw = (2 - i)(-1 + i) = -2 + 2i + i - i^2 = -2 + 3i - (-1)$
$= -1 + 3i.$
(d) $\frac{1}{w} = \frac{1\overline{w}}{w\overline{w}} = \frac{1(-1 - i)}{(-1 + i)(-1 - i)} = \frac{-1 - i}{1 - i^2} = \frac{-1 - i}{1 - (-1)} = -\frac{1}{2} - \frac{1}{2}i.$

The points are shown in Figure 8. ◄

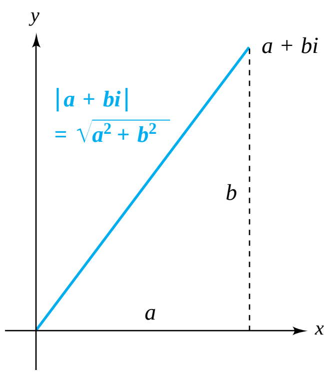


FIGURE 9
The absolute value of a complex number is the distance to the origin in the complex plane.

Absolute Value of a Complex Number

We define the absolute value of a real number x as the *distance* between the point x and the origin 0. In a similar manner, we define the absolute value of $a + bi$ as the distance in the plane between (a, b) and the origin $(0, 0)$. The diagram in Figure 9 and the Pythagorean theorem give a distance of $\sqrt{a^2 + b^2}$.

Definition: absolute value of a complex number

Suppose z is the complex number $a + bi$. The *absolute value of z*, denoted by $|z|$, is $\sqrt{a^2 + b^2}$, and we write $|z| = \sqrt{a^2 + b^2}$.

Many of the properties of absolute values of complex numbers are the same as those for real numbers. From the definition of distance in the next section, we can also observe that $|z - w|$ is the distance between the complex numbers z and w.

Properties of absolute value of a complex number

If z and w are any complex numbers, then:

1. $|z| \geq 0$

2. $|\bar{z}| = |-z| = |z|$

3. $|z \cdot w| = |z| \cdot |w|$

4. $\sqrt{z \cdot \bar{z}} = |z|$

5. $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$ if $w \neq 0$

6. $|z + w| \leq |z| + |w|$

►EXAMPLE 4 ***Absolute values of complex numbers*** Suppose $z = 2 - i$ and $w = -1 + i$ (the complex numbers of Example 3). Verify that the properties of absolute values hold for each pair.

(a) $|z| + |w|, |z + w|$ **(b)** $|\bar{z}|, |-z|$ **(c)** $|zw|, |z| \cdot |w|$

Solution

(a) $|z| + |w| = |2 - i| + |-1 + i| = \sqrt{2^2 + (-1)^2} + \sqrt{(-1)^2 + 1^2}$
$= \sqrt{5} + \sqrt{2}.$

$|z + w| = |1 + 0i| = \sqrt{1^2 + 0^2} = 1.$

Since $1 < \sqrt{5} + \sqrt{2}$, $|z + w| < |z| + |w|$ (Property 6).

(b) $|\bar{z}| = |2 + i| = \sqrt{2^2 + 1^2} = \sqrt{5}$;
$|-z| = |-2 + i| = \sqrt{(-2)^2 + 1^2} = \sqrt{5}.$

Thus $|\bar{z}| = |-z|$ (Property 2).

(c) $zw = -1 + 3i$, and so $|zw| = |-1 + 3i| = \sqrt{10}$.
$|z| \cdot |w| = \sqrt{5} \cdot \sqrt{2} = \sqrt{10}$. Therefore $|zw| = |z| \cdot |w|$ (Property 3). ◄

Ordering of the Complex Numbers

Real numbers are ordered by the less than relation, so that if b is any real number, then exactly one of the following is true:

$$b = 0 \quad \text{or} \quad b < 0 \quad \text{or} \quad b > 0.$$

Since R is a subset of C, any ordering of the complex numbers should be consistent with the ordering of R. Consider the nonzero complex number i. If we could extend the ordering of R to C, then we would have to have

$$i < 0 \quad \text{or} \quad i > 0.$$

If $i > 0$, then we can multiply both sides by i and get

$$i \cdot i > i \cdot 0 \quad \text{or} \quad -1 > 0,$$

which is not a true statement in R. That leaves only the possibility that i is negative, $i < 0$. If we multiply both sides by a negative number, we must reverse the direction of the inequality, and we get the same contradiction:

$$i \cdot i > i \cdot 0 \quad \text{or} \quad -1 > 0.$$

We are forced to conclude that *there is no consistent way to order the set of complex numbers using the less than relation.* Thus, for example, we cannot say that one of the two numbers $3 - 4i$ and $-1 + 2i$ is less than the other.

EXERCISES 1.3

Check Your Understanding

Exercises 1–5 True or False. Give reasons.

1. If both b and d are negative real numbers, then $\sqrt{-b}\sqrt{-d} = \sqrt{bd}$.

2. $|3 - \sqrt{10}| = \sqrt{10} - 3$.

3. $1 - \sqrt{10} > 1 - \pi$

4. In the complex plane, $3 + 4i$ is farther from the origin than $5i$.

5. If $0 < b < 1$ and $-1 < c < 0$, then $-1 < b - c < 1$.

Exercises 6–10 Fill in the blank so that the resulting statement is true.

6. The smallest integer that is greater than $3 - \sqrt{10}$ is ______.

7. The greatest integer less than $\frac{3-\pi}{4}$ is ______.

8. The largest integer in the set $\{x \mid -\sqrt{3} < x < \sqrt{148}\}$ is ______.

9. The number of integers between $\sqrt{29} + 1$ and 8π is ______.

10. The number of prime numbers between $\sqrt{17} - 2$ and $\sqrt{83} + 8$ is ______.

Develop Mastery

Exercises 1–6 ***Number Line*** *Show the set on a number line.*

1. $\{x \mid x > -2 \text{ and } x < 2\}$

2. $\{x \mid x \geq -1 \text{ and } x \leq 1\}$

3. $\{x \mid x < 1 \text{ or } x > 3\}$

4. $\{x \mid x \leq -1\} \cup \{x \mid x > 4\}$

5. $\{x \mid x > 0\} \cap \{x \mid x < 3\}$

6. $\{x \mid 1 \leq x < \sqrt{7}\}$

Exercises 7–9 ***Absolute Value*** *Simplify. Express in exact form without using absolute values, and as a decimal approximation rounded off to four decimal places.*

7. (a) $\left|\frac{1}{4} - \frac{3}{2}\right|$ **(b)** $\left|3 - \frac{9}{2}\right|$

8. (a) $\left|1 - \frac{4}{7}\right|$ **(b)** $|3 - \sqrt{17}|$

9. (a) $|\pi - 3|$ **(b)** $\left|\pi - \frac{22}{7}\right|$

Exercises 10–13 Enter one of the three symbols $<$, $>$, or $=$ in each blank space to make the resulting statement true.

10. (a) -4 ______ -6
(b) $-\pi$ ______ $-\sqrt{10}$

11. (a) $\frac{5}{11}$ ______ 0.45
(b) $1 + \sqrt{2}$ ______ 2.9

12. (a) $\frac{47}{3}$ ______ 16
(b) $0.\overline{63}$ ______ $\frac{7}{11}$

13. (a) $|1 - \sqrt{3}|$ ______ $\sqrt{3} - 1$
(b) $|-5|$ ______ 4

Exercises 14–17 ***Ordering Numbers*** *Order the set of three numbers from smallest to largest. Express the result using the symbol $<$, as, for instance, $y < z < x$.*

14. $x = 5, y = -7, z = -3$

15. $x = \frac{16}{23}, y = \frac{5}{12}, z = \frac{7}{15}$

16. $x = 1 - \sqrt{3}, y = \sqrt{3} - 1, z = -1$

17. $x = \left|1 - \frac{7}{5}\right|, y = \left|1 - \frac{6}{5}\right|, z = \left|1 - \frac{1}{5}\right|$

Exercises 18–19 True or False.

18. (a) $\pi^2 < 10$ **(b)** $\frac{1}{\sqrt{2} - 1} > 2.28$

19. (a) $1.33 < 1.\overline{3}$ **(b)** $0.54 > \frac{6}{11}$

Exercises 20–25 ***Intervals on Number Line*** *Show the intervals on a number line.*

20. $(-1, 4)$

21. $(-\infty, 2)$

22. $[-2, \infty)$

23. $[1, 4] \cap (0, 5)$

24. $(-\infty, 3) \cup (3, 4]$

25. $[-3, 2] \cap [2, 5]$

Exercises 26–31 ***Verbal to Number Line*** *Set S is described verbally. Show S on the number line.*

26. Set S contains all negative numbers greater than -5.

27. Set S contains all real numbers greater than -2 and less than 3.

28. Set S consists of all integers between -3 and 8.

29. Set S consists of all prime numbers between 0 and 16.

30. Set S consists of all real numbers between $-\sqrt{3}$ and $\sqrt{5}$.

31. Set S consists of all positive real numbers less than 4.

32. What is the largest integer that is **(a)** less than or equal to -5? **(b)** less than -5?

33. What is the largest integer that is less than $1 + \sqrt{17}$?

34. What is the smallest integer that is greater than $\frac{348}{37}$?

35. What is the smallest even integer that is greater than $12 + \sqrt{5}$?

36. What is the largest prime number that is less than $\frac{23}{0.23}$?

Exercises 37–50 ***Complex Number Arithmetic*** *Perform the indicated operations. Express the result as a complex number in standard form.*

37. $(5 + 2i) + (3 - 6i)$ **38.** $(3 - i) + (-1 + 5i)$

39. $(6 - i) - (3 - 4i)$ **40.** $8 - (3 + 5i) + 2i$

41. $(2 + i)(3 - i)$ **42.** $(-1 + i)(2 + 3i)$

43. $(1 + 3i)(1 - 3i)$ **44.** $(7 - 2i)(7 + 2i)$

45. $\dfrac{1 + 3i}{i}$ **46.** $\dfrac{1 + i}{1 - i}$

47. $(1 + \sqrt{3}i)^2$ **48.** $\dfrac{2i}{(1 - i)(2 - i)}$

49. **(a)** i^2 **(b)** i^6 **(c)** i^{12} **(d)** i^{18}

50. **(a)** i^5 **(b)** i^9 **(c)** i^{15} **(d)** i^{21}

Exercises 51–55 If z is $1 - i$ and w is $-2 + i$, express in standard form.

51. $z + 3w$ **52.** $zw - 4$ **53.** $\overline{z} \cdot \overline{w}$

54. $|z + w|$ **55.** $\dfrac{z - \overline{w}}{w}$

Exercises 56–59 ***Complex Plane*** *For the given z and w, show in the complex plane:*

(a) z ***(b)*** w ***(c)*** $\overline{z}$ ***(d)*** $z + w$ ***(e)*** $z \cdot w$

56. $z = 2 - 2i$; $w = 3 + 4i$

57. $z = -3 + 2i$; $w = -2 - i$

58. $z = -1 + 2i$; $w = 3i$

59. $z = 5 - i$; $w = -1 + i$

60. **(a)** If $z = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$; find z^2, z^3, z^4, z^5, z^6.

(b) Evaluate each of $|z|, |z^2|, |z^3|, \ldots, |z^6|$.

61. From Exercise 60 draw a diagram showing z, z^2, z^3, z^4, z^5, and z^6 in the complex plane. Note the distance from each of these points to the origin. On what circle do these points lie?

Exercises 62–63 In Exercises 60–61, replace z with $z = \frac{1}{\sqrt{2}}(1 + i)$.

1.4 RECTANGULAR COORDINATES, TECHNOLOGY, AND GRAPHS

Creative people live in two worlds. One is the ordinary world which they share with others and in which they are not in any special way set apart from their fellow men. The other is private and it is in this world that the creative acts take place. It is a world with its own passions, elations and despairs, and it is here that, if one is as great as Einstein, one may even hear the voice of God.

Mark Kac

Rectangular Coordinates

Few intellectual discoveries have had more far-reaching consequences than coordinatizing the plane by René Descartes nearly 400 years ago. We speak of **Cartesian** or rectangular coordinates in his honor.

A rectangular coordinate system uses two perpendicular number lines in the plane, which we call coordinate axes. The more common orientation is a horizontal ***x*-axis** and a vertical ***y*-axis,** but other variable names and orientations are sometimes useful.

Each point P in the plane is identified by an ordered pair of real numbers (c, d), called the **coordinates** of P, where c and d are numbers on the respective axes as shown in Figure 10. Conversely, every pair of real numbers names a unique point on the plane.

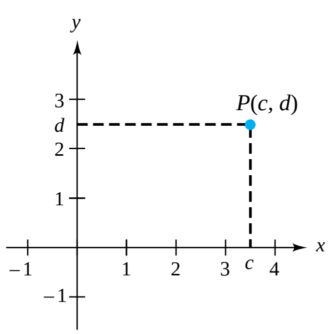


FIGURE 10

A rectangular system of coordinates provides a one-to-one correspondence between the set of ordered pairs of real numbers and the points in the plane.